Modal Testing

(Lecture 21)

Dr. Hamid Ahmadian

School of Mechanical Engineering

Iran University of Science and Technology

ahmadian@iust.ac.ir

Derivation of Mathematical Models

- Output error method
  - Eigen-sensitivity method
    - Inverse Eigensensitivity Method (6.3.7)
  - FRF-sensitivity method
    - Response Function Method (6.3.8)
- Bounds of errors in parameter estimation
- Homework 4

Output error method

- The updating is performed by minimizing the difference between the actual response and the predicted one.

$$\begin{Bmatrix} \Delta\omega_1^2 \\ \Delta\omega_2^2 \\ \vdots \\ \{\Delta\phi_1\} \\ \{\Delta\phi_2\} \\ \vdots \end{Bmatrix} = \begin{bmatrix} \frac{\partial\omega_1^2}{\partial p_1} & \frac{\partial\omega_1^2}{\partial p_2} & \frac{\partial\omega_1^2}{\partial p_3} & \cdots \\ \frac{\partial\omega_2^2}{\partial p_1} & \frac{\partial\omega_2^2}{\partial p_2} & \frac{\partial\omega_2^2}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \frac{\partial\{\Delta\phi_1\}}{\partial p_1} & \frac{\partial\{\Delta\phi_1\}}{\partial p_2} & \frac{\partial\{\Delta\phi_1\}}{\partial p_3} & \cdots \\ \frac{\partial\{\Delta\phi_2\}}{\partial p_1} & \frac{\partial\{\Delta\phi_2\}}{\partial p_2} & \frac{\partial\{\Delta\phi_2\}}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \vdots \end{Bmatrix}$$

Modal Sensitivities

$$\left([K]-\omega_r^2[M]\right)\{\phi_r\}=\{0\},$$

$$\frac{\partial}{\partial p}\left([K]-\omega_r^2[M]\right)\{\phi_r\}=\{0\},$$

$$\left([K]-\omega_r^2[M]\right)\frac{\partial\{\phi_r\}}{\partial p}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\},$$

Eigenvalue Sensitivity

$$Multiply\,by \quad \{\phi_r\}^T$$

$$\{\phi_r\}^T\left([K]-\omega_r^2[M]\right)\frac{\partial\{\phi_r\}}{\partial p}+$$

$$\{\phi_r\}^T\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\},$$

$$results \Rightarrow \frac{\partial\omega_r^2}{\partial p}=\frac{\{\phi_r\}^T\left(\frac{\partial[K]}{\partial p}-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}}{\{\phi_r\}^T[M]\{\phi_r\}}$$

Eigenvector Sensitivity

$$Starting \quad from:$$

$$\left(\left[K\right]-\omega_r^2\left[M\right]\right)\frac{\partial\left\{\phi_r\right\}}{\partial p}+\left(\frac{\partial\left[K\right]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}\left[M\right]-\omega_r^2\frac{\partial\left[M\right]}{\partial p}\right)\left\{\phi_r\right\}=\left\{0\right\},$$

$$and\ taking \quad \frac{\partial\left\{\phi_r\right\}}{\partial p}=\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_{rj}\left\{\phi_j\right\}$$

$$\Rightarrow\left(\left[K\right]-\omega_r^2\left[M\right]\right)\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_{rj}\left\{\phi_j\right\}+\left(\frac{\partial\left[K\right]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}\left[M\right]-\omega_r^2\frac{\partial\left[M\right]}{\partial p}\right)\left\{\phi_r\right\}=\left\{0\right\}$$

Eigenvector Sensitivity

$$\left([K]-\omega_r^2[M]\right)\sum_{\substack{j=1\\j\neq r}}^{N}\gamma_{rj}\{\phi_j\}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

$$\Rightarrow\{\phi_s\}^T\left([K]-\omega_r^2[M]\right)\sum_{\substack{j=1\\j\neq r}}^{N}\gamma_{rj}\{\phi_j\}+\{\phi_s\}^T\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

$$\Rightarrow\left(\omega_s^2-\omega_r^2\right)\gamma_{rs}+\{\phi_s\}^T\left(\frac{\partial[K]}{\partial p}-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

Eigenvector Sensitivity

$$\Rightarrow \left(\omega_s^2 - \omega_r^2\right)\gamma_{rs} + \{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\} = 0,$$

$$\Rightarrow \gamma_{rs} = \frac{\{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\}}{\left(\omega_r^2 - \omega_s^2\right)}$$

$$\Rightarrow \quad \frac{\partial \{\phi_r\}}{\partial p} = \sum_{\substack{s=1 \\ s \neq r}}^{N} \frac{\{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\}}{\left(\omega_r^2 - \omega_s^2\right)} \{\phi_s\}$$

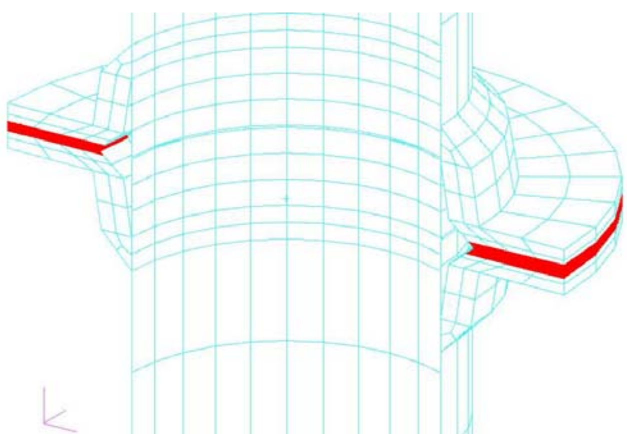
Updating, Redesign, Reanalysis

$$
\begin{Bmatrix} \Delta\omega_1^2 \\ \Delta\omega_2^2 \\ \vdots \\ \{\Delta\phi_1\} \\ \{\Delta\phi_2\} \\ \vdots \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial\omega_1^2}{\partial p_1} & \dfrac{\partial\omega_1^2}{\partial p_2} & \dfrac{\partial\omega_1^2}{\partial p_3} & \cdots \\ \dfrac{\partial\omega_2^2}{\partial p_1} & \dfrac{\partial\omega_2^2}{\partial p_2} & \dfrac{\partial\omega_2^2}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \dfrac{\partial\{\Delta\phi_1\}}{\partial p_1} & \dfrac{\partial\{\Delta\phi_1\}}{\partial p_2} & \dfrac{\partial\{\Delta\phi_1\}}{\partial p_3} & \cdots \\ \dfrac{\partial\{\Delta\phi_2\}}{\partial p_1} & \dfrac{\partial\{\Delta\phi_2\}}{\partial p_2} & \dfrac{\partial\{\Delta\phi_2\}}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \vdots \end{Bmatrix}
$$

BOLTED JOINT MODELS

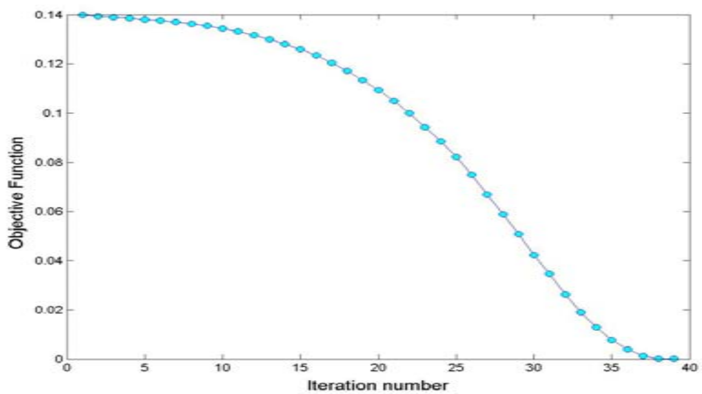
- In face-to-face contacts the behavior of the joint is governed mainly by normal stiffness and shear stiffness

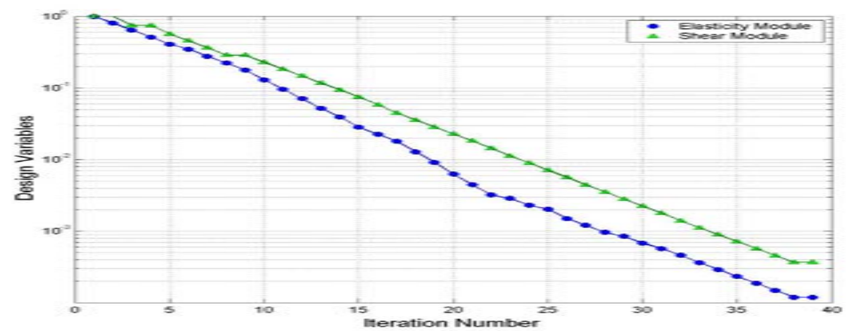
$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}=\begin{bmatrix}\lambda+2G & \lambda & \lambda & & & \\ & \lambda+2G & \lambda & & & \\ & & \lambda+2G & & & \\ & & & G & & \\ & & & & G & \\ sym & & & & & G\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix}, \qquad \lambda=\frac{G(E-2G)}{(E-3G)}$$

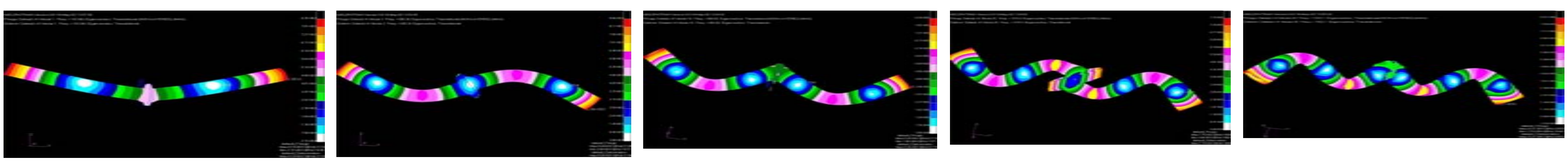


MODEL IDENTIFICATION

- The updating was performed using the *Design Sensitivity Module* available in MSC/NASTRAN 2001**.**

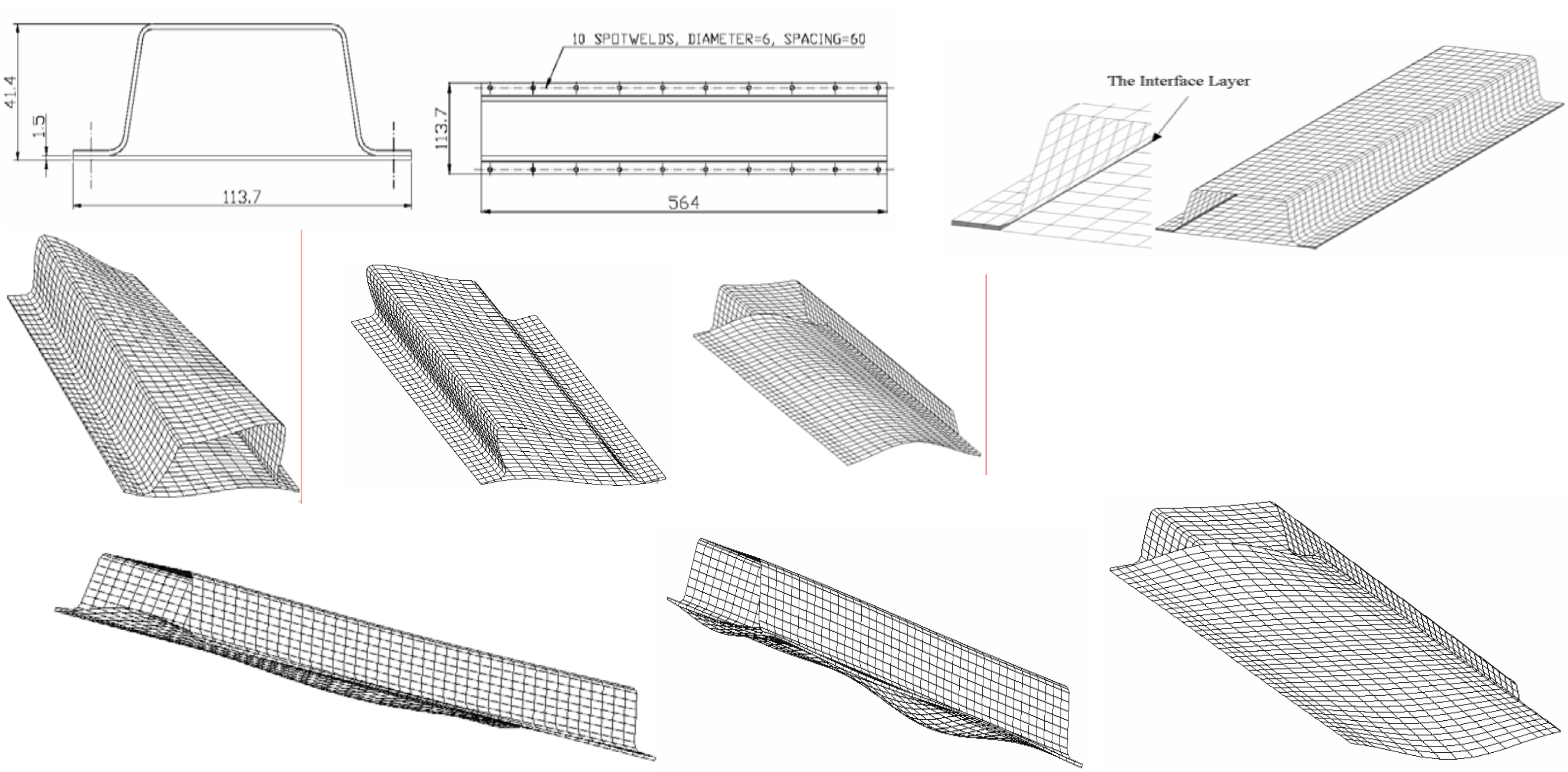






MODELLING OF SPOT WELDS

- Uncertainty in car body modeling:



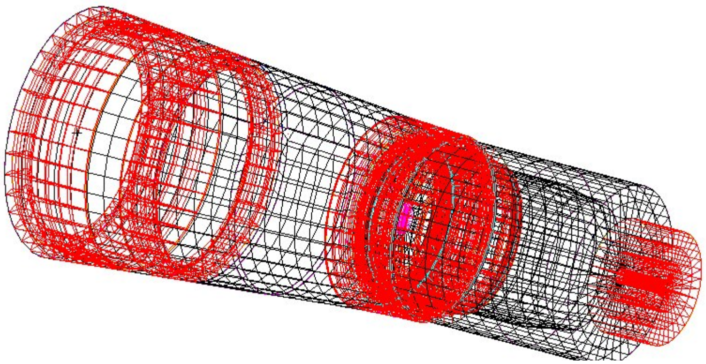
MODELLING OF SPOT WELDS

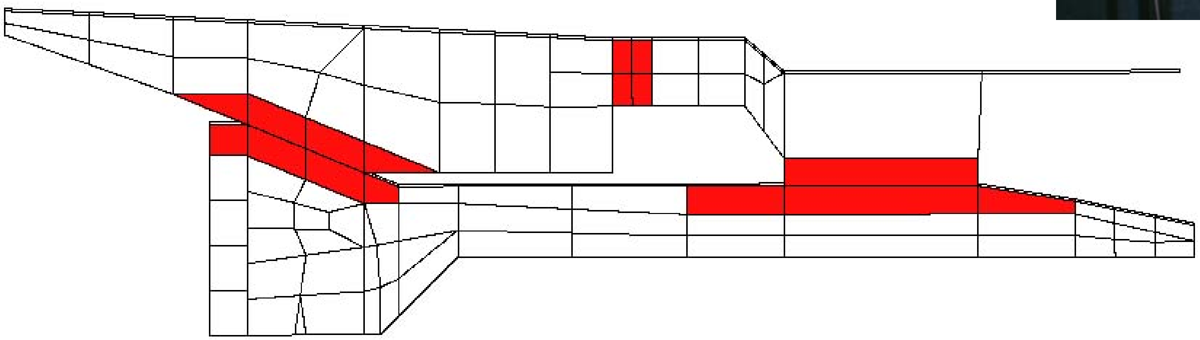
- The updating results:

No.	Measured	Updated	Error
1	537.3	531.2	-1.1
2	574.8	582.0	1.2
3	629.4	616.4	-2.0
4	664.4	668.3	0.5
5	672.2	669.6	-0.3
6	701.2	677.9	-3.3
7	734.4	734.6	0.02
8	821.4	813.6	-0.9
9	865.1	865.0	-0.01
10	946.4	908.7	-3.9

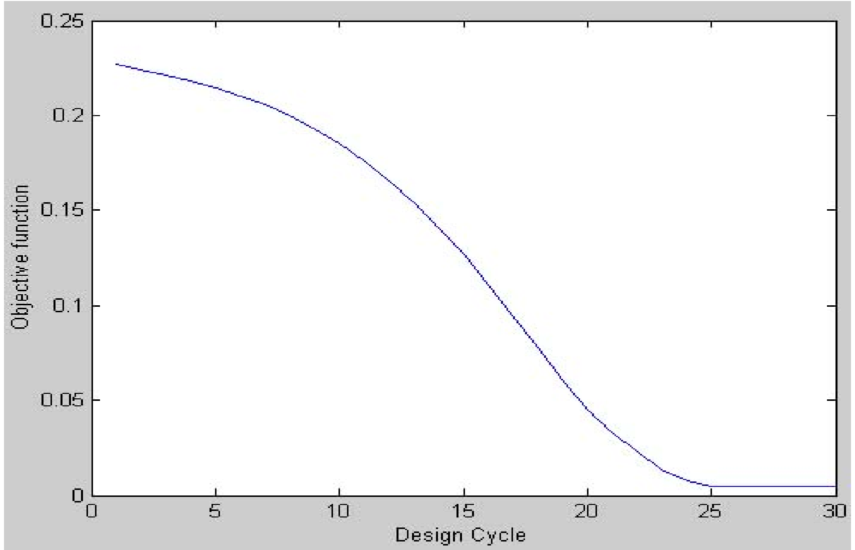
THE MACE PROJECT

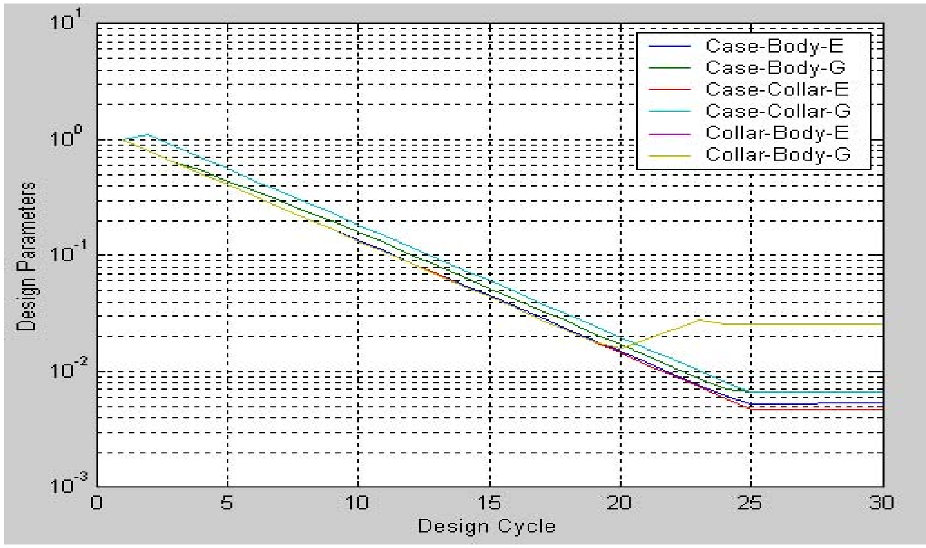
- AWE case study project





UPDATING RESULTS





UPDATING RESULTS

Mode No.	Test (Hz)	FEM (Hz)	Error (%)	Updated	Error (%)
1	551	777	41	546	-0.9
2	612	635	3.75	622	1.63
Torsional mode	N/A	1161		1018	
Axial mode	N/A	1285		1079	
3	1119	1186	5.98	1125	0.53
4	1175	1163	-1.02	1177	0.17
5	1337	1415	5.83	1334	-0.22
6	1516	1643	8.37	1604	5.8
7	1645	1848	12.34	1687	2.55
8	1717	1761	2.56	1744	1.57

FRF Sensitivities

$$[Z(\omega)] = [K] + i\omega[C] - \omega^2[M],$$

$$\Rightarrow ([A]+[B])^{-1} = [A]^{-1} - ([A]+[B])^{-1}[B][A]^{-1}$$

$$take\,[A] \Rightarrow \quad [Z(\omega)]_A, \qquad\qquad [A+B] \Rightarrow \quad [Z(\omega)]_x$$

$$then \Rightarrow [Z(\omega)]_x^{-1} = [Z(\omega)]_A^{-1} - [Z(\omega)]_x^{-1}([Z(\omega)]_x - [Z(\omega)]_A)^{-1}[Z(\omega)]_A^{-1}$$

$$[\alpha(\omega)]_x - [\alpha(\omega)]_A = -[\alpha(\omega)]_x[\Delta Z(\omega)][\alpha(\omega)]_A$$

FRF Sensitivities

$$\left[\alpha(\omega)\right]_x - \left[\alpha(\omega)\right]_A = -\left[\alpha(\omega)\right]_x \left[\Delta Z(\omega)\right]\left[\alpha(\omega)\right]_A,$$

$$\left\{\alpha_x(\omega) - \alpha_A(\omega)\right\}_j^T = \left\{\alpha_x(\omega)\right\}_j^T \left[\Delta Z(\omega)\right]\left[\alpha(\omega)\right]_A$$

FRF Sensitivities

$$\frac{\partial[\alpha(\omega)]}{\partial p}=\frac{\partial\left([Z(\omega)]^{-1}\right)}{\partial p}=-[Z(\omega)]^{-1}\frac{\partial[Z(\omega)]}{\partial p}[Z(\omega)]^{-1}$$

$$\frac{\partial[\alpha(\omega)]}{\partial p}=-[\alpha(\omega)]\frac{\partial[Z(\omega)]}{\partial p}[\alpha(\omega)]$$

$$\frac{\partial[\alpha(\omega)]}{\partial p}=-[\alpha(\omega)]\left(\frac{\partial[K]}{\partial p}+i\omega\frac{\partial[C]}{\partial p}-\omega^2\frac{\partial[M]}{\partial p}\right)[\alpha(\omega)]$$

Bounds of errors in parameter estimation

$$Linear \quad eqns:$$

$$Ax = b$$

$$Pertubation \quad of \; A$$

$$(A + \Delta A)(x + \Delta x) = b$$

$$A\Delta x + \Delta A x + O(\Delta^2) = 0.$$

$$\Delta x \approx -A^{-1}\Delta A x$$

Bounds of errors in parameter estimation

$$\|\Delta x\| = \|A^{-1}\Delta A x\| \le \|A^{-1}\| \cdot \|\Delta A x\| \le \|A^{-1}\| \cdot \|\Delta A\| \cdot \|x\|$$

$$\Rightarrow \frac{\|\Delta x\|}{\|x\|} \le \|A^{-1}\| \cdot \|A\| \cdot \frac{\|\Delta A\|}{\|A\|}$$

$$\Rightarrow \frac{\|\Delta x\|}{\|x\|} = \kappa(A) \cdot \frac{\|\Delta A\|}{\|A\|}$$

Bounds of errors in parameter estimation

$$Perturbation \quad of\ b:$$

$$A(x+\Delta x) = b + \Delta b \qquad A\Delta x = \Delta b \Rightarrow \Delta x = A^{-1}\Delta b$$

$$\|\Delta x\| \le \|A^{-1}\| \cdot \|\Delta b\|$$

$$\frac{\|\Delta x\|}{\|A\| \cdot \|x\|} \le \frac{\|\Delta x\|}{\|Ax\|} \le \|A^{-1}\| \frac{\|\Delta b\|}{\|b\|}$$

$$\frac{\|\Delta x\|}{\|x\|} \le \|A^{-1}\| \cdot \|A\| \cdot \frac{\|\Delta b\|}{\|b\|} \Rightarrow \frac{\|\Delta x\|}{\|x\|} \le \kappa(A) \frac{\|\Delta b\|}{\|b\|}$$

Home Work 4

- Develop a procedure to locate a crack in a simply supported damped beam using output error strategy;
  - Using eigen-sensitivity method
  - Using FRF sensitivity method
  - The system is structurally damped:
    - non-proportional localized to the crack
    - The stiffness matrix is complex

Modal Testing

(Lecture 21-1)

Dr. Hamid Ahmadian

School of Mechanical Engineering

Iran University of Science and Technology

ahmadian@iust.ac.ir

Derivation of Mathematical Models

- Model Updating
  - To fine-tune some parameters to minimize the discrepancy between the model predictions and the measured data.
- Model Parameterization
  - Matrix updating
  - Physical parameter updating
  - Generic element models

Introduction

- Finite element model updating is employed to bring the predictions of the model into agreement with experimental observations from a physical structure.
- This can be achieved provided that the measured data represent the actual behavior of the structure.
- Then accuracy of the updated model depends upon the parameters chosen for updating.

Introduction

- There are basically two parameter selection strategies in the literature.
  - One approach is to select the geometric or material input data of the finite element model
  - The second strategy, in a contrast to the first, allows changes in all entries of the system matrices or a subset of them.

Introduction

- The first approach is very popular:
  - it can be implemented in existing finite element codes
  - there is a readily available physical explanation for each modified term.
- But it has some drawbacks as well

Introduction

- The method is incapable of changing the mathematical ‘‘structure’’ of the model.
- Structural mis-modelling and omitted effects cannot be corrected.
- Errors of this type include
  - the omission of shear effects,
  - stress stiffening and coupling of bending and torsion in beams.

Introduction

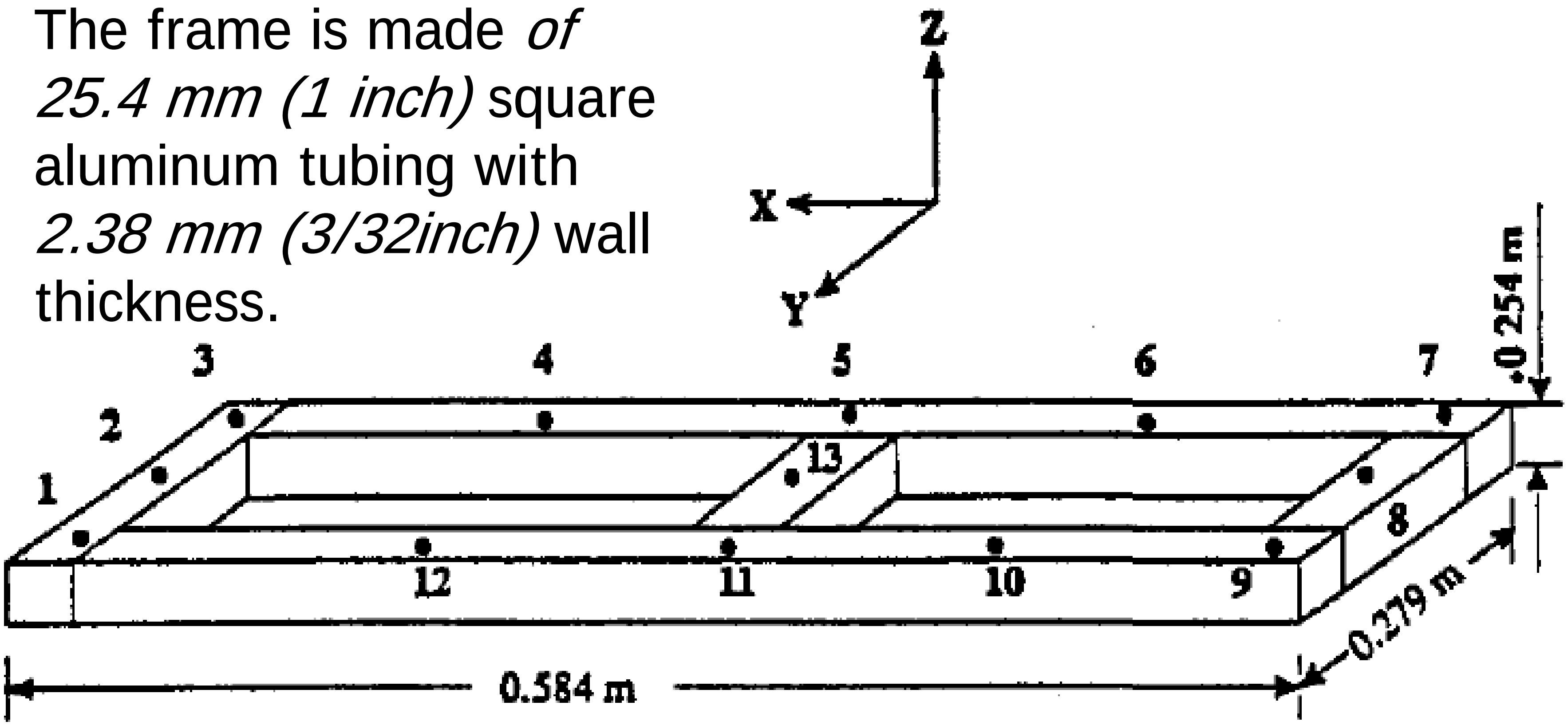
- The second strategy, allows the updated model to reproduce observed behavior exactly.
- But there is no guarantee that it represents a physical system and not a meaningless numerical expression that reproduces the test data.
- A common problem is the loss of positivity of system matrices.

Performance of Updating Procedures

- In this section we update the stiffness matrix of the frame structure using the various methods:
  - Matrix updating;
  - Matrix updating maintaining the pattern of zeros in the model;
  - Physical parameter updating;
  - Using generic stiffness matrices.

Frame structure and measured coordinates

The frame is made *of 25.4 mm (1 inch)* square aluminum tubing with *2.38 mm (3/32inch)* wall thickness.



The Finite Element Model

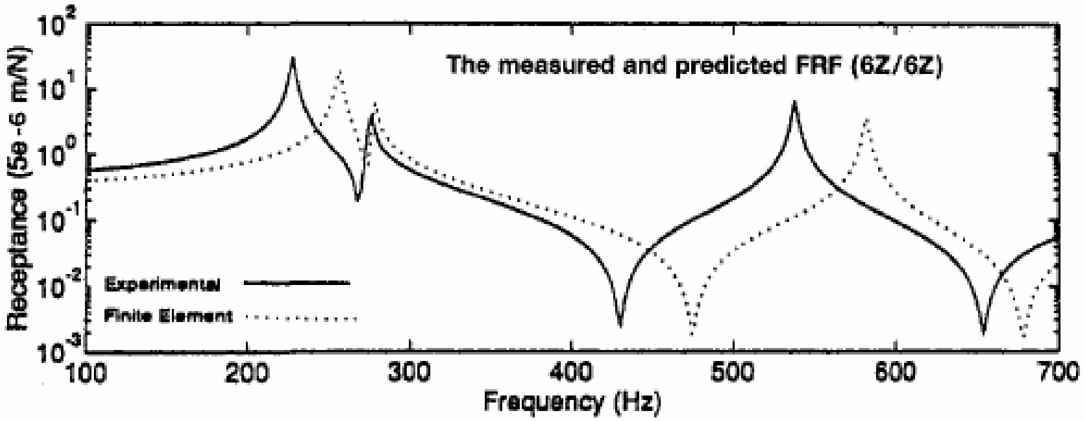
- Consists of 28 in-plane frame elements (combination of a beam element and a rod element).
- The beam part is modeled using Euler-Bernoulli beam theory.
- The displacement vector of the element is:

$$\left[w_{i-1},\, L\frac{dw_{i-1}}{dx},\, L\theta_{i-1},\, w_i,\, L\frac{dw_i}{dx},\, L\theta_i \right]$$

Discrepancy of the FE and test results

Computed and measured natural frequencies

Mode No.	Natural frequency (HZ) FE model	Measured	error %
1	255.8	226.8	12.8
2	277.5	275.2	0.9
3	581.3	537.4	8.3
4	911.3	861.5	6.0
5	1049.4	974.8	8.0



Expanding the mode shapes

$$\min_{\phi_i^{(2)}} \left\| (K_0 - \lambda_i \mathbf{M}_0) \begin{Bmatrix} \boldsymbol{\phi}_i^{(1)} \\ \boldsymbol{\phi}_i^{(2)} \end{Bmatrix} \right\| \quad i = 1, \ldots, 5$$

$$\boldsymbol{\Phi}^T \mathbf{M}_0 \boldsymbol{\Phi} = \begin{bmatrix} 1 & -0.0221 & 0.0207 & 0.0133 & 0.0105 \\ & 1 & 0.0036 & -0.0001 & -0.0096 \\ & & 1 & -0.0171 & -0.0022 \\ & \mathit{Sym.} & & 1 & 0.0064 \\ & & & & 1 \end{bmatrix}$$

Expanding the mode shapes

- We interpreted the near-orthogonality of the modes with respect to $\mathbf{M_o}$ as evidence that
  - our measurements were accurate, and
  - $\mathbf{M_o}$ adequately represented the mass matrix of the structure.
- Having ascertained that $\mathbf{M_o}$ was adequate we extended them so that the extended modes would be precisely orthogonal with respect to $\mathbf{M_o}$

Performance of Updating Procedures

- The model parameters are adjusted by forming an equation error function using the first three quasi-measured modes.
- We judge the performance of each method by:
  - Its ability to reproduce the first three measured modes;
  - To predict the fourth and fifth measured modes;
  - More importantly, by its ability to predict the modes of the structure when there is a design change.

Performance of Updating Procedures

- We identify the model of the test structure by an iterative procedure in which each iteration has two sub-steps:
  - use the current estimate of ***K***, along with $\boldsymbol{M_o}$ to obtain Φ
  - use the obtained Φ to compute a new estimate of ***K***, using the analysis described before.

Matrix Updating Method

Baruch and Bar Itzhack(1978)

$$min \, \|\mathbf{M}_0^{-1/2}(\mathbf{K}-\mathbf{K}_0)\mathbf{M}_0^{-1/2}\|,$$

$$subject\ to \quad \mathbf{K\Phi}=\mathbf{M}_0\mathbf{\Phi\Lambda}, \quad \mathbf{\Phi}^T\mathbf{K\Phi}=\mathbf{\Lambda}, \quad and \quad \mathbf{K}=\mathbf{K}^T$$

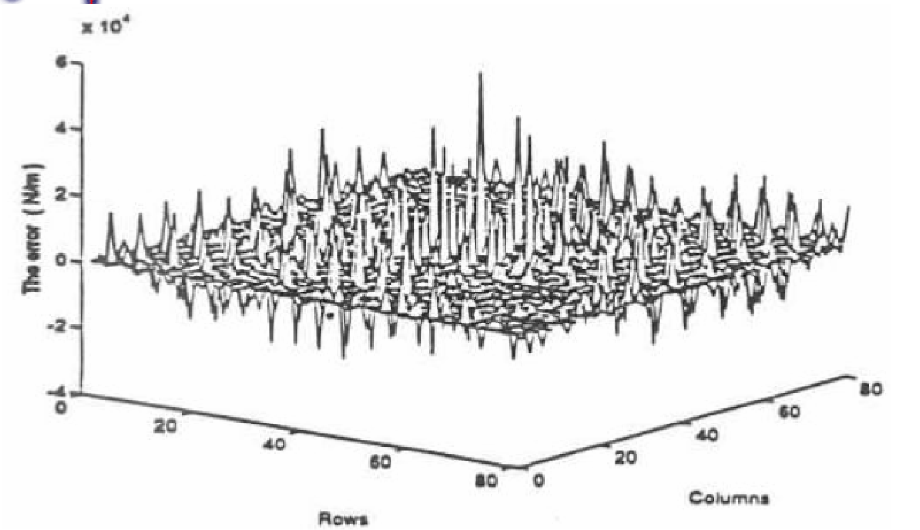
The final equation in the procedure is a closed form solution for the updated stiffness matrix:

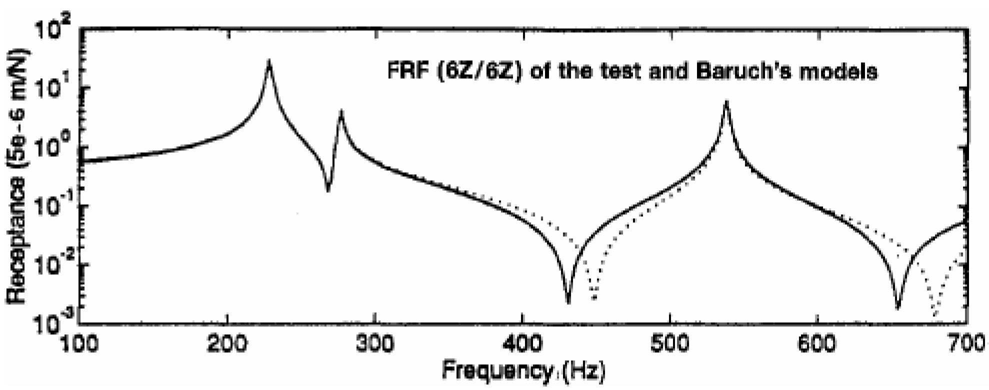
$$\mathbf{K}=\mathbf{K}_0+\mathbf{\Delta}+\mathbf{\Delta}^T,$$

$$\mathbf{\Delta}=(\mathbf{I}-\mathbf{M}_0\mathbf{\Phi\Phi}^T/2)(\mathbf{M}_0\mathbf{\Phi\Lambda}-\mathbf{K}_0\mathbf{\Phi})\mathbf{\Phi}^T\mathbf{M}_0.$$

Matrix Updating Method

Baruch and Bar Itzhack(1978)





Mode No.	Predicted by FEM	Baruch's Method	Measured
Rigid	0	0	0
body	0	0	0
modes	0	0	0
1	255.8	226.8	226.8
2	277.5	275.2	275.2
3	581.3	537.4	537.4
4	911.3	911.3	861.5
5	1049.4	1049.4	974.8

Matrix Updating Method
Baruch and Bar Itzhack(1978)

- We notice that except for the modes used in updating, Baruch's model has the same eigen-data as the original finite element model.

$$
\begin{aligned}
\|\mathbf{M}_0^{-1/2}(\mathbf{K}-\mathbf{K}_0)\mathbf{M}_0^{-1/2}\|
&= \|\mathbf{M}_0^{-1/2}\Big(\sum_{i=1}^{N}\lambda_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T - \lambda_{0i}\boldsymbol{\phi}_{0i}\boldsymbol{\phi}_{0i}^T\Big)\mathbf{M}_0^{-1/2}\| \\
&= \|\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T - \mathbf{V}_0\boldsymbol{\Lambda}_0\mathbf{V}_0^T\|
\end{aligned}
$$

$$\mathbf{v}_i = \mathbf{M}_0^{1/2}\boldsymbol{\phi}_i,$$

$$\mathbf{v}_{0i} = \mathbf{M}_0^{1/2}\boldsymbol{\phi}_{0i},$$

Matrix Updating Method

Baruch and Bar Itzhack(1978)

$$\mathbf{V} = \mathbf{V}_0 \begin{bmatrix} \mathbf{R}_m & 0 \\ 0 & \mathbf{Q}_{N-m} \end{bmatrix},$$

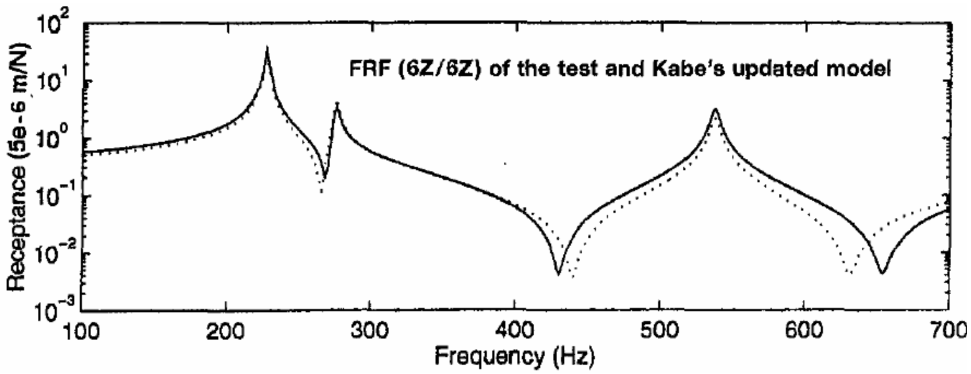
$$min \left\| \mathbf{V}_0 \begin{bmatrix} \mathbf{R}_m \Lambda_m \mathbf{R}_m^T - \Lambda_{0_m} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{N-m} \Lambda_{N-m} \mathbf{Q}_{N-m}^T - \Lambda_{0_{N-m}} \end{bmatrix} \mathbf{V}_0^T \right\|.$$

$$\mathbf{Q}_{N-m} = \mathbf{I}_{N-m}$$

$$\Lambda_{N-m} = \Lambda_{0_{N-m}}.$$

The updated model is consistent with the test results, and beyond that its eigendata is the same as that of the finite element model

Matrix updating maintaining the pattern of zeros Kabe (1985)



Mode No.	Predicted by FEM	Changing non-zeros	Changing non-zeros + rigid modes	Measured
Rigid	0	1197.6 *i*	0	0
body	0	456.8 *i*	0	0
modes	0	206.7	0	0
1	255.8	226.8	226.8	226.8
2	277.5	275.2	275.2	275.2
3	581.3	537.4	537.4	537.4
4	911.3	548.3	862.6	861.5
5	1049.4	652.0	897.6	974.8

Physical Parameter Updating

- Parameters are *EI/L*^{3}, *GJ/L*
- The updated model has the correct definiteness properties, but is little better than the original FE model in predicting the measured frequencies.

Mode No.	Predicted by FEM	Physical Parameter	Measured
Rigid	0	0	0
body	0	0	0
modes	0	0	0
1	255.8	255.6	226.8
2	277.5	277.4	275.2
3	581.3	580.1	537.4
4	911.3	911.6	861.5
5	1049.4	1043.2	974.8

GENERIC ELEMENT MATRICES

- Basic assumption in every updating procedure is that the order and the structure of the finite element model is correct.
- A generic element model is built by imposing all necessary conditions that the element must satisfy.

GENERIC ELEMENT MATRICES

- Necessary conditions:
  - **M** is positive definite,
  - **K** is semi-positive definite.

$$\mathbf{K}\mathbf{\Phi}_R = 0, \quad \mathbf{\Phi}_R^T \mathbf{M} \mathbf{\Phi}_R = \begin{bmatrix} \mathbf{m} & 0 \\ 0 & \mathbf{J} \end{bmatrix}$$

  - Geometric symmetry

$$\mathbf{K} = \mathbf{R}^T \mathbf{K} \mathbf{R}, \quad \mathbf{M} = \mathbf{R}^T \mathbf{M} \mathbf{R}$$

A generic beam element

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} & k_{23} & k_{24} \\ & & k_{33} & k_{34} \\ Sym. & & & k_{44} \end{bmatrix}$$

$$T^{\mathrm{T}}KT = K, \quad T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} & -k_{14} & k_{24} \\ & & k_{11} & -k_{12} \\ Sym. & & & k_{22} \end{bmatrix}$$

→ **Six independent parameters**

A generic beam element

$$K\Phi = 0, \quad \Phi = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1/2 & 1 & 1/2 & 1 \end{bmatrix}^T \longrightarrow \mathbf{K} = \begin{bmatrix} k_{ww} & k_{ww}/2 & -k_{ww} & k_{ww}/2 \\ & k_{\theta\theta} & -k_{ww}/2 & k_{ww}/2 - k_{\theta\theta} \\ & & k_{ww} & -k_{ww}/2 \\ Sym. & & & k_{\theta\theta} \end{bmatrix},$$

$$k_{\theta\theta} > k_{ww}/4 > 0,$$

$$\mathbf{M} = \rho AL \begin{bmatrix} m_{11} & m_{12} & \frac{1}{2} - m_{11} & m_{14} \\ & m_{22} & -m_{14} & m_{24} \\ & & m_{11} & -m_{12} \\ Sym. & & & m_{22} \end{bmatrix},$$

$$m_{24} = \frac{1}{6} - \frac{m_{11}}{2} + m_{12} + m_{14} - m_{22},$$

A generic beam element

Euler-Bernoulli beam model $k_{ww}=12EI,\ \ k_{\theta\theta}=4EI$

Timoshenko beam element $k_{ww}=EI\dfrac{12}{1+g}\quad k_{\theta\theta}=EI\dfrac{4+g}{1+g}\quad g=\dfrac{CEI}{GAL^2}$

A beam element with a crack

$$k_{ww}=\frac{12EI}{1+(1-\nu^2)\alpha^3F_2},$$

$$k_{\theta\theta}=\frac{EI[4+(1-\nu^2)(18\alpha F_1+2\alpha^3F_2)]}{[1+6(1-\nu^2)\alpha F_1][1+2(1-\nu^2)\alpha^3F_2]}$$

Generic Frame Stiffness Matrices

- Updating the stiffness matrix of the frame by modifying its eigendata.
- Each element stiffness matrix has order six and rank three:
  - The strain modes occupy the same range as their FE counterparts.
  - Symmetry of element can be preserved in modal domain.

Generic Frame Stiffness Matrices

- In general, it may be defined using six parameters:

$$\mathbf{K}^e = \mathbf{U}_0 \mathbf{R} \Lambda \mathbf{R}^T \mathbf{U}_0^T = \mathbf{U}_0 \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ & & k_{33} \end{bmatrix} \mathbf{U}_0^T,$$

where

$$\mathbf{U}_0^T = \begin{bmatrix} 0 & \alpha & 0 & 0 & -\alpha & 0 \\ 2\beta & \beta & 0 & -2\beta & \beta & 0 \\ 0 & 0 & \alpha & 0 & 0 & -\alpha \end{bmatrix}, \quad \begin{aligned} \alpha &= \sqrt{2}/2 \\ \beta &= \sqrt{10}/10 \end{aligned}$$

Generic Frame Stiffness Matrices

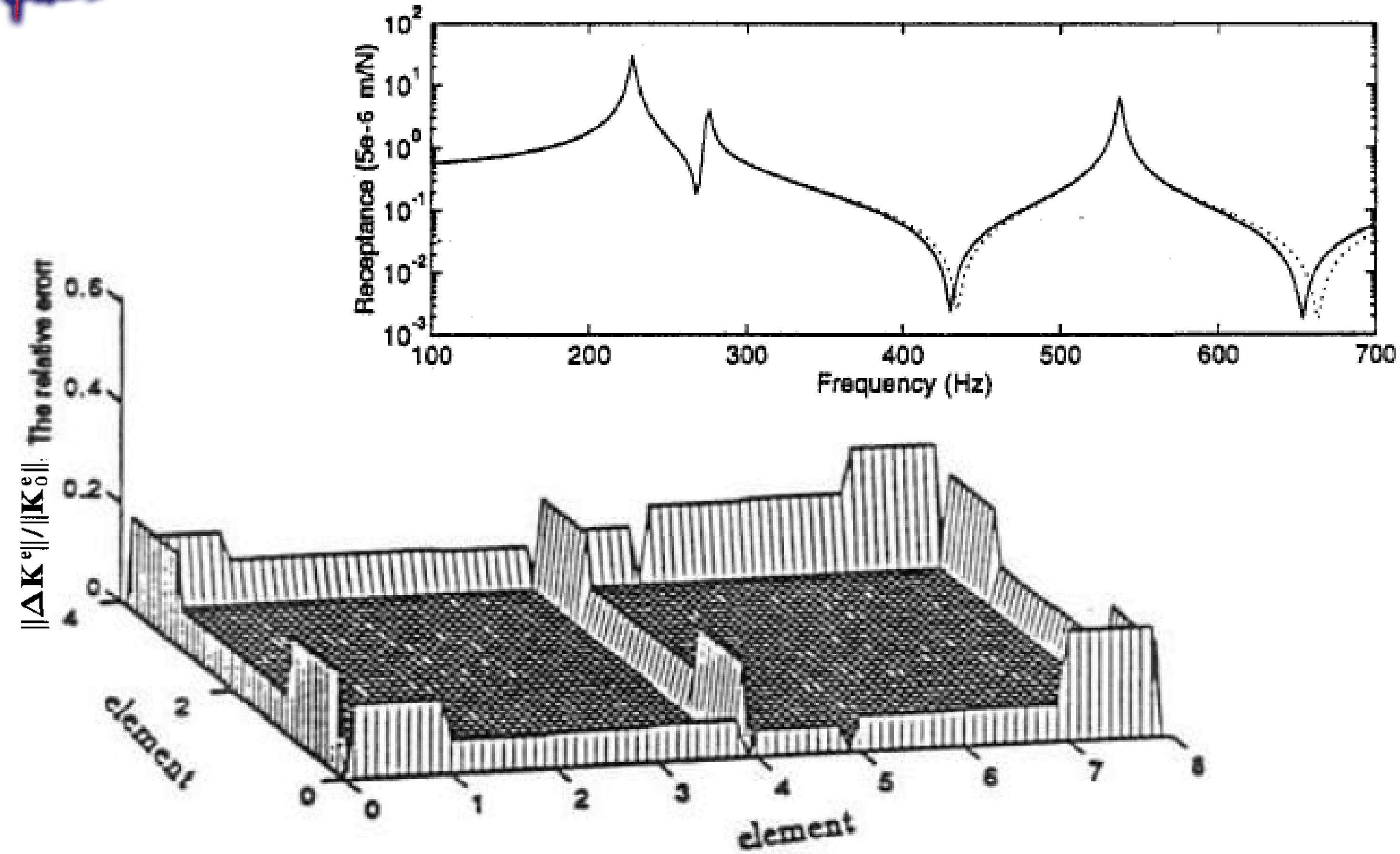
- The diagonal terms k_{11} , k_{22} and k_{33} represent, respectively, the effects of bending, shear and twisting modes in the element,
- The off diagonal terms, k_{12} k_{13} k_{23} account for the coupling effects between these modes.
- The first strain mode of the element is symmetric, while the second and third modes are antisymmetric.
- Thus for any symmetrical frame element, i.e. not a joint element, k_{12} *and* k_{13} must be zero.

Generic Frame Stiffness Matrices

Mode No.	Predicted by FEM	LS Solution	Adjusted Solution	Measured
Rigid	0	0	0	0
body	0	0	0	0
modes	0	0	0	0
1	255.8	226.8	226.8	226.8
2	277.5	275.2	275.2	275.2
3	581.3	537.4	537.4	537.4
4	911.3	861.9	862.5	861.5
5	1049.4	918.8	968.3	974.8

By requiring similar elements have similar models,

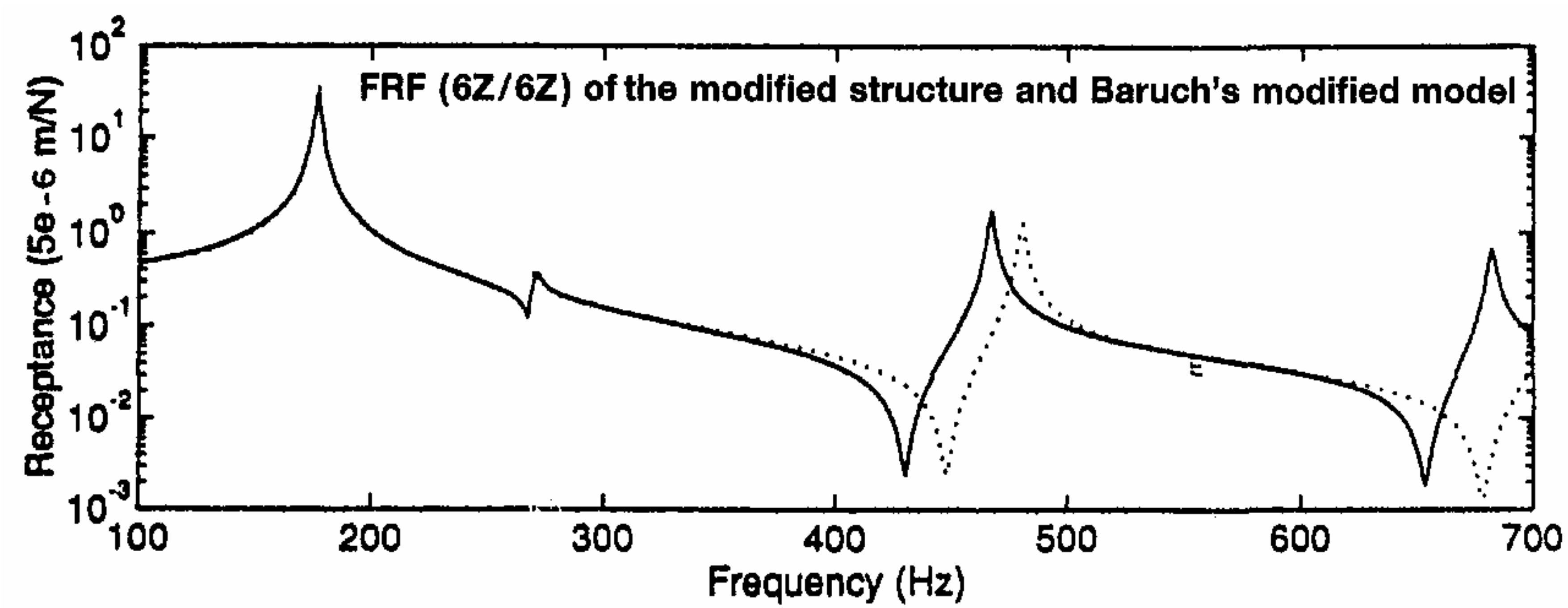
Generic Frame Stiffness Matrices



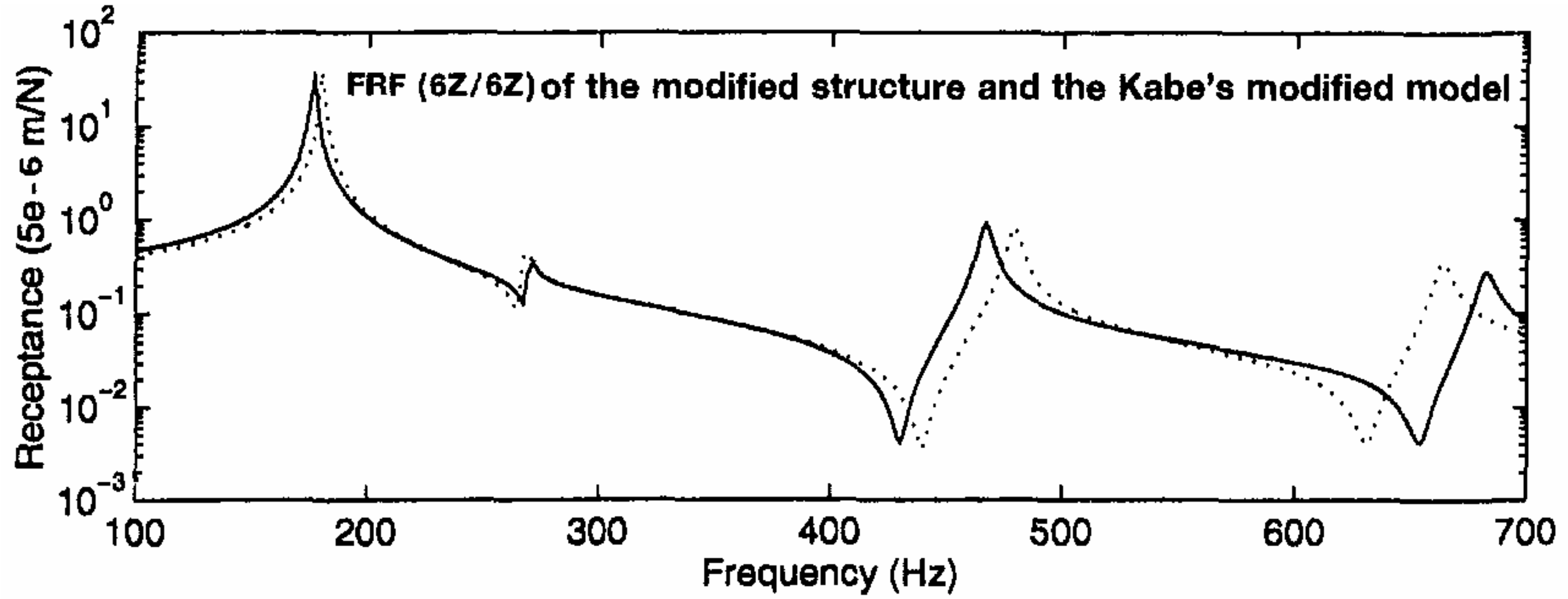
Introducing a design change

- Adding a lumped mass at coordinate 6 and grounding the structure from this coordinate using a spring.
- This modification shifts the fourth mode of the structure below *700 Hz.*
- Followings show the predictions of different models superimposed on the modified structure response.

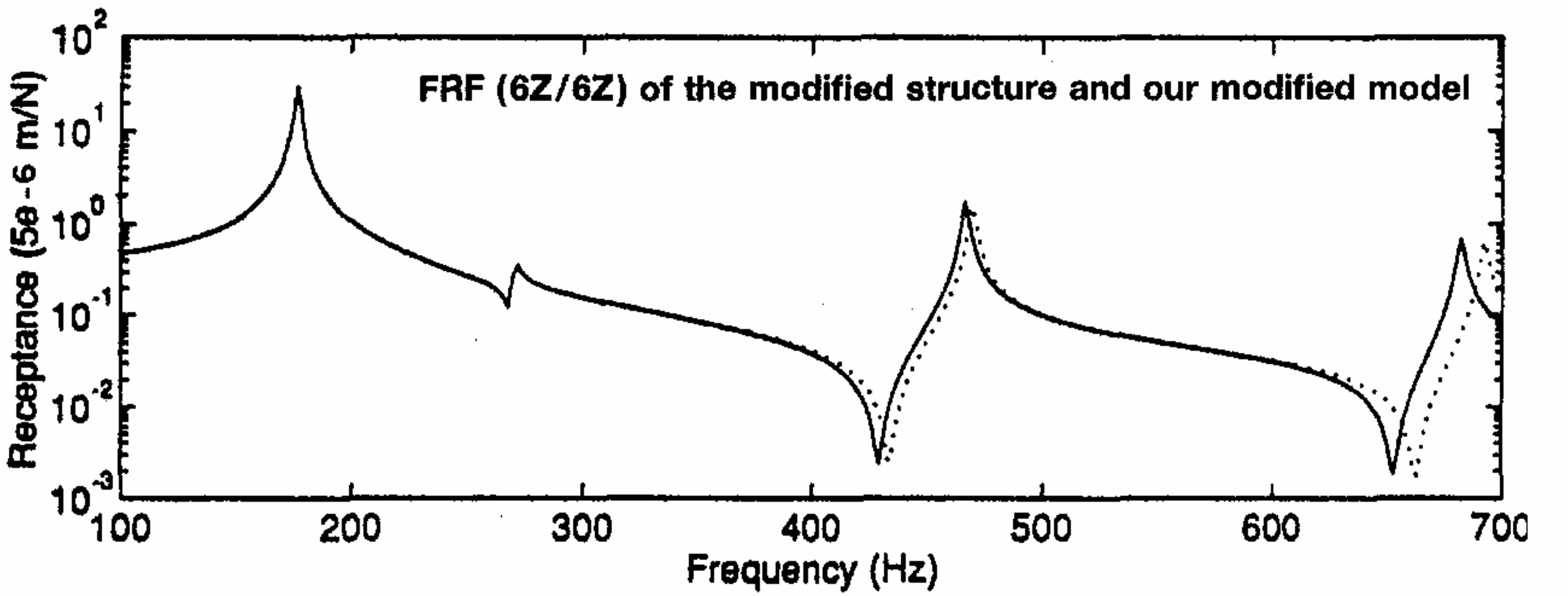
Introducing a design change



Introducing a design change



Introducing a design change



Conclusion

- The success of updating procedures depended on the way the model parameters are selected.
- Updating the model by adjusting all the (non-zero) entries yields a model consistent with the test data, but the model may not correspond to a physical structure.
- Adjusting only the physical parameters does not produce a model consistent with the test data.
- The answer appears to lie in defining a generic model for each element and minimizing the error function by adjusting the acceptable model parameters.

Modal Testing

(Lecture 22)

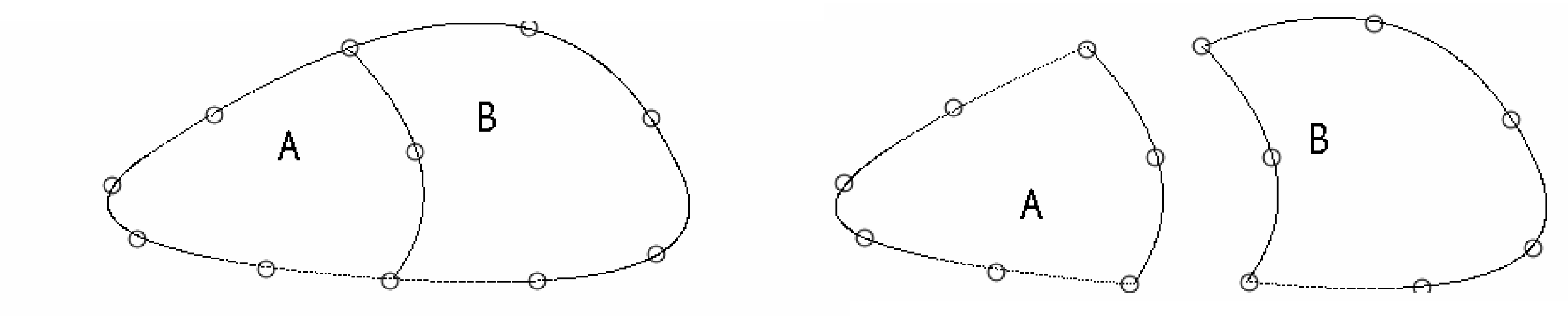
Dr. Hamid Ahmadian

School of Mechanical Engineering

Iran University of Science and Technology

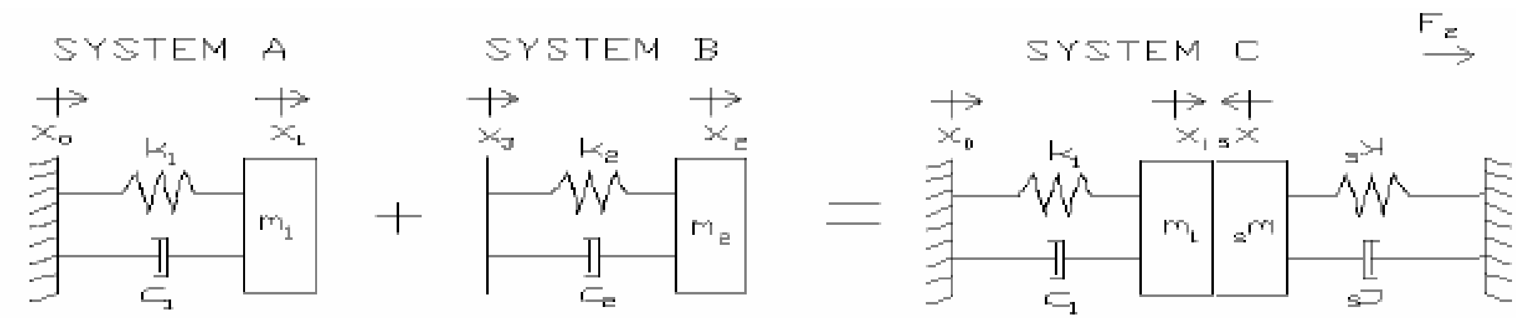
ahmadian@iust.ac.ir

Coupled & Modified Structure Analysis



- Coupled & Modified Structure Analysis (Section 6.4)
  - Structural Modifications
  - Coupled Structures
  - Sub-structuring

FRF Methods of Coupled Structure Analysis



$$X_A = H_A(\omega)\ F_A, \qquad X_C = X_B = X_A,$$

$$X_B = H_B(\omega)\ F_B. \qquad F_C = F_B + F_A.$$

$$H_C^{-1} = H_B^{-1} + H_A^{-1} = Z_A + Z_B$$

FRF Methods of Coupled Structure Analysis

- Extention to the case where several DOFs involved in the coupling process,
  - No other DOFs are included in the analysis

$$H_C^{-1} = H_B^{-1} + H_A^{-1}, \quad H_C^{-1} = H_A^{-1}\left(I + H_A H_B^{-1}\right),$$

$$H_C^{-1} = H_A^{-1}\left(H_B + H_A\right)H_B^{-1},$$

$$H_C = H_B\left(H_B + H_A\right)^{-1} H_A.$$

A more efficient formula from the numerical viewpoint.

FRF Methods of Coupled Structure Analysis

- An appropriate form for modification applications:

$$H_C = H_B\left(H_B + H_A\right)^{-1} H_A,$$

$$H_C = \left(H_A + H_B - H_A\right)\left(H_B + H_A\right)^{-1} H_A$$

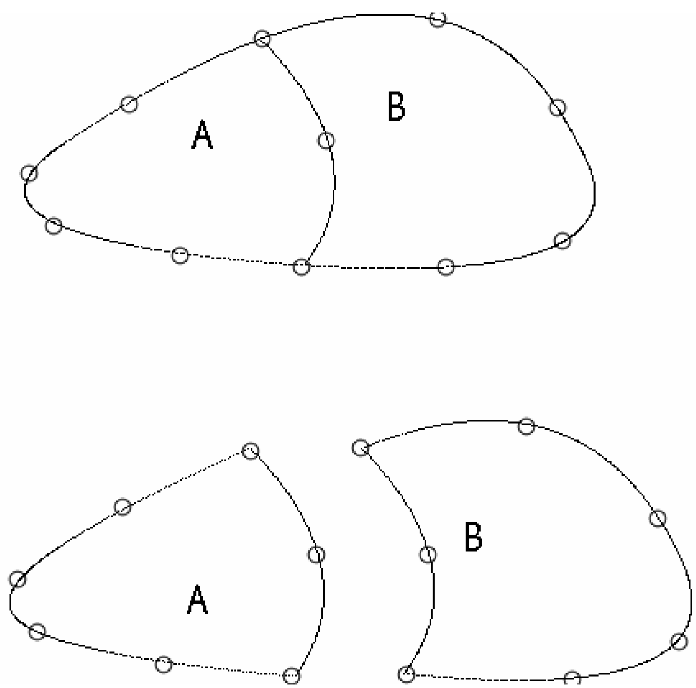
$$H_C = H_A - H_A\left(H_B + H_A\right)^{-1} H_A.$$

FRF Methods of Coupled Structure Analysis

- The general case:

$$Z_A = H_A^{-1} = \begin{bmatrix} Z_{\alpha\alpha}^A & Z_{\alpha c}^A \\ Z_{c\alpha}^A & Z_{cc}^A \end{bmatrix},$$

$$Z_B = H_B^{-1} = \begin{bmatrix} Z_{\beta\beta}^B & Z_{\beta c}^B \\ Z_{c\beta}^B & Z_{cc}^B \end{bmatrix},$$



FRF Methods of Coupled Structure Analysis

$$H_C^{-1} = H_B^{-1} \oplus H_A^{-1}$$

$$Z_C = Z_A \oplus Z_B$$

$$Z_C = \begin{bmatrix} Z_{\alpha\alpha}^A & 0 & Z_{\alpha c}^A \\ 0 & Z_{\beta\beta}^B & Z_{\beta c}^B \\ Z_{c\alpha}^A & Z_{c\beta}^B & Z_{cc}^A + Z_{cc}^B \end{bmatrix}$$

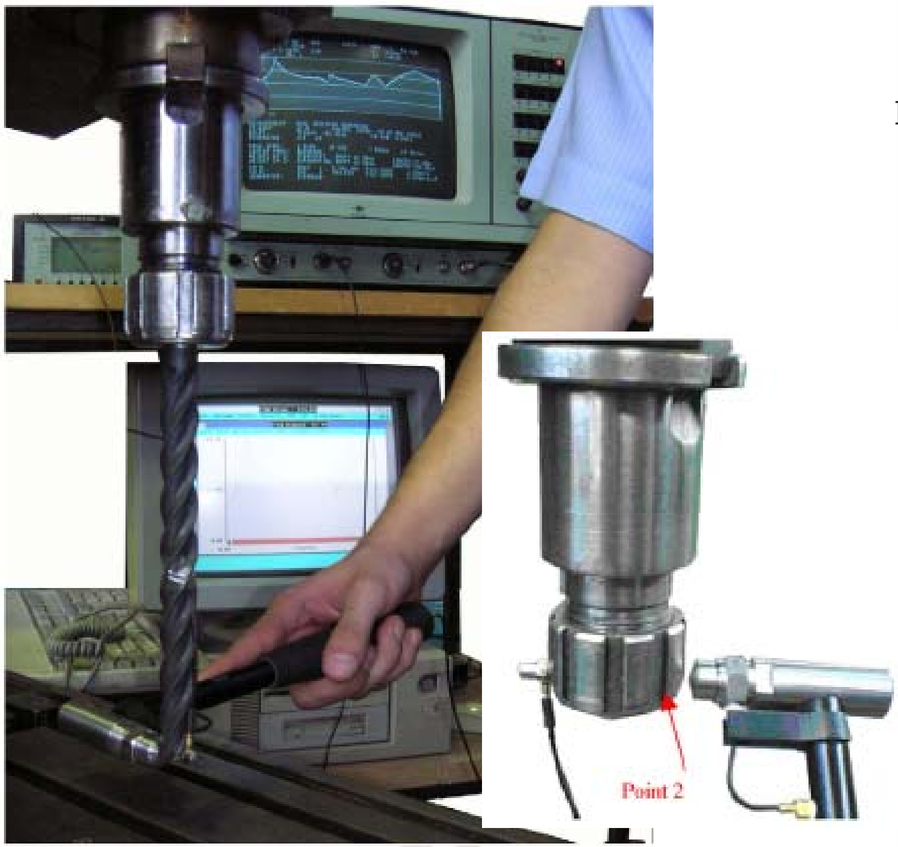
FRF Methods of Coupled Structure Analysis

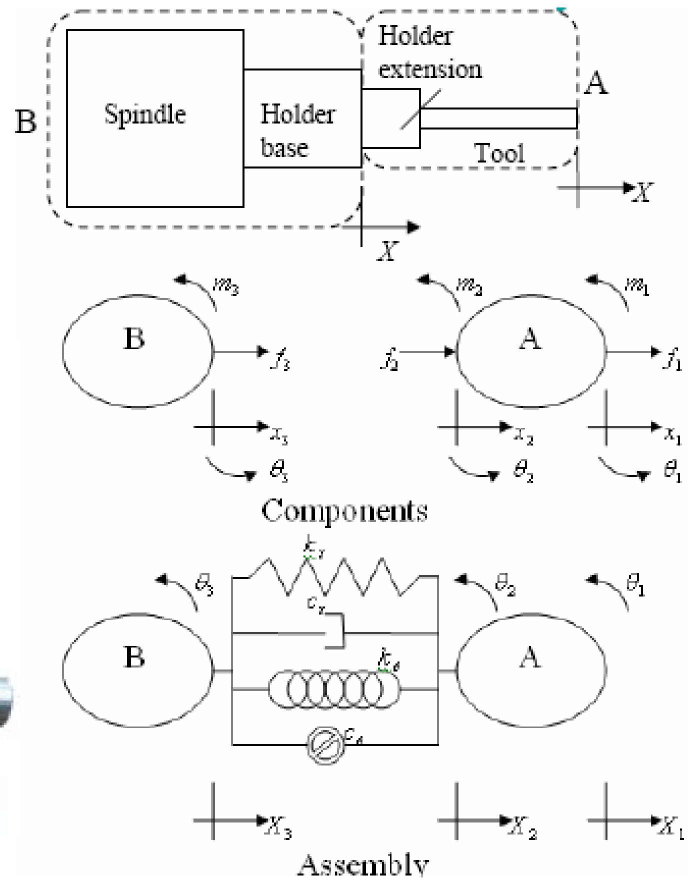
$$H_C = H_A - H_A\left(H_B + H_A\right)^{-1} H_A.$$

$$H_C = \begin{bmatrix} H^A_{\alpha\alpha} & H^A_{\alpha c} & 0 \\ H^A_{c\alpha} & H^A_{cc} & 0 \\ 0 & 0 & H^B_{\beta\beta} \end{bmatrix}$$

$$- \begin{Bmatrix} H^A_{\alpha c} \\ H^A_{cc} \\ -H^B_{\beta c} \end{Bmatrix} \left[H^A_{cc} + H^B_{cc} \right]^{-1} \left\{ H^A_{c\alpha} \quad H^A_{cc} \quad -H^B_{c\beta} \right\}$$

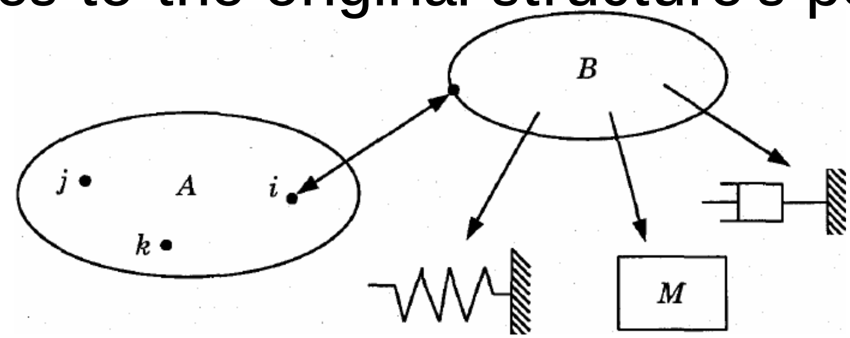
Case Study


Point 2

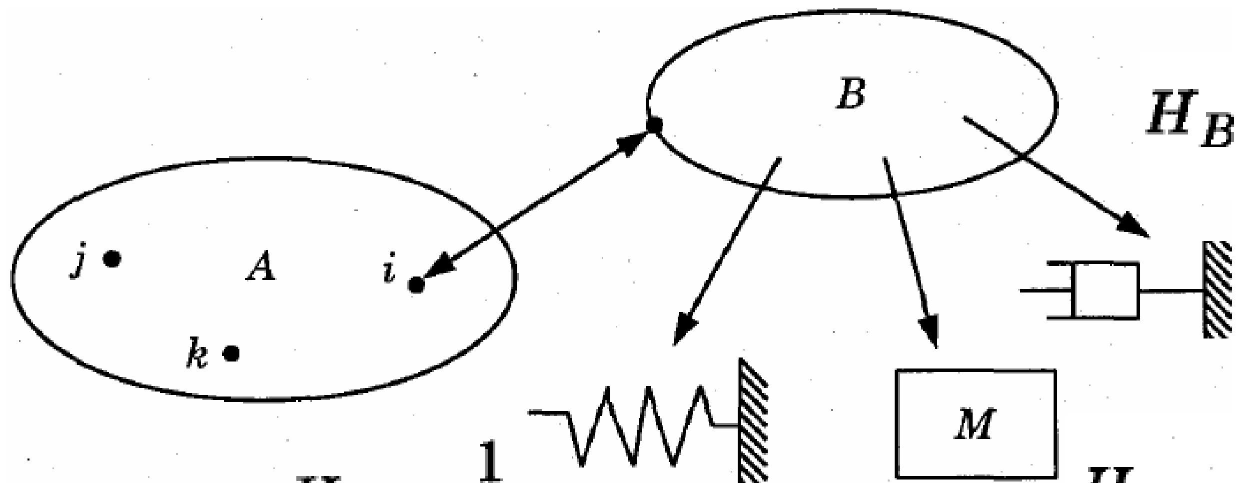

Holder
extension
A
B
Spindle
Holder
base
Tool
X
X
B
A
Components
B
A
Assembly

Simplified Expressions for SDOF Connections

- What will be the changes to the structure's dynamic properties if a specific modification is applied at a given point?
- These situations tend to be concerned with:
  - Applications of relatively simple modifications
  - To identify the best places to introduce modifications in order to bring about desired changes to the original structure's performance.



Simplified Expressions for SDOF Connections



$$H_B = \frac{1}{i\omega c_B}$$

$$H_B = \frac{1}{k_B}$$

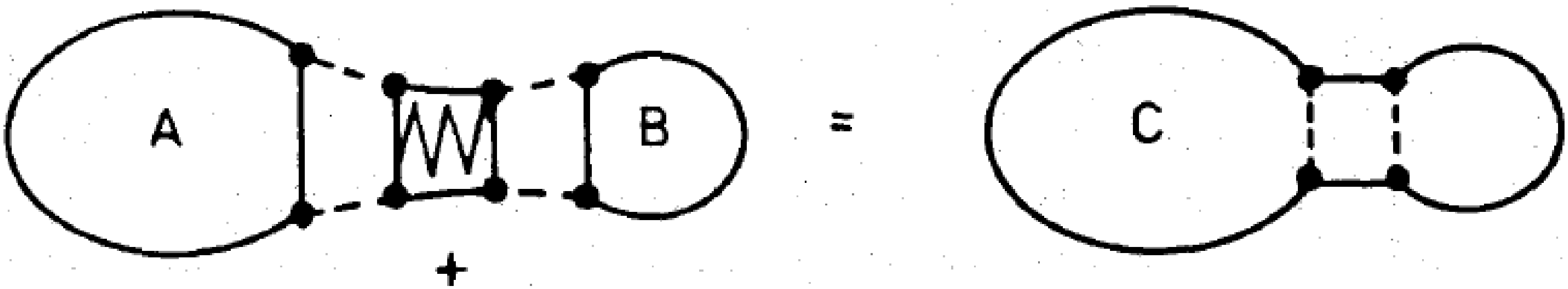
$$H_B = \frac{1}{-\omega^2 m_B}$$

$$H_B = \frac{-\left(k + i\omega c - \omega^2 m\right)}{\omega^2 m\left(k + i\omega c\right)}$$

for the mass - spring - damper absorber subsystem

$$H_{jk}^{A'} = H_{jk}^{A} - H_{ij}^{A} H_{ik}^{A}\left(H_{ii}^{A} + H_B\right)^{-1}$$

Modal Analysis of Coupled and Modified Structures



$$[M_A]_{N_A \times N_A}\{\ddot{x}_A\} + [K_A]\{x_A\} = \{f_A\}$$
$$[M_B]_{N_B \times N_B}\{\ddot{x}_B\} + [K_B]\{x_B\} = \{f_B\}$$

Forces present at the connection DOFs

$$[I]_{m_A \times m_A}\{\ddot{p}_A\} + \lfloor \omega_A^2 \rfloor \{p_A\} = [\Phi_A]^T_{m_A \times m_A}\{f_A\}$$
$$[I]_{m_B \times m_B}\{\ddot{p}_B\} + \lfloor \omega_B^2 \rfloor \{p_B\} = [\Phi_B]^T_{m_B \times m_B}\{f_B\}$$

$$m_A, m_B < N_A, N_B$$

Modal Analysis of Coupled and Modified Structures

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{p}_A \\ \ddot{p}_B \end{Bmatrix} + \begin{bmatrix} \omega_A^2 & 0 \\ 0 & \omega_B^2 \end{bmatrix} \begin{Bmatrix} p_A \\ p_B \end{Bmatrix} = \begin{bmatrix} \Phi_A^T & 0 \\ 0 & \Phi_B^T \end{bmatrix} \begin{Bmatrix} f_A \\ f_B \end{Bmatrix}$$

$$\begin{Bmatrix} f_A \\ f_B \end{Bmatrix}_{(n_A+n_B)\times 1} = [K_{CC}] \begin{Bmatrix} x_A \\ x_B \end{Bmatrix} = [K_{CC}] \begin{bmatrix} [\Phi_A] & 0 \\ 0 & [\Phi_B] \end{bmatrix} \begin{Bmatrix} p_A \\ p_B \end{Bmatrix}$$

$$[I] \begin{Bmatrix} \ddot{p}_A \\ \ddot{p}_B \end{Bmatrix} + \left(\begin{bmatrix} \Phi_A^T & 0 \\ 0 & \Phi_B^T \end{bmatrix} [K_{CC}] \begin{bmatrix} \Phi_A & 0 \\ 0 & \Phi_B \end{bmatrix} + \begin{bmatrix} \omega_A^2 & 0 \\ 0 & \omega_B^2 \end{bmatrix} \right) \begin{Bmatrix} p_A \\ p_B \end{Bmatrix} = \{0\}$$

Modal Analysis of Coupled and Modified Structures

- One of main drawbacks of this approach is the exclusion of the higher modes,
  - the modal truncation problem
- The effect of out-of-range high-frequency modes can be approximated by residual terms which are essentially damped springs.