Modal Testing

(Lecture 11)

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Response Function Measurement Techniques

- Introduction
- Test Planning
- Basic Measurement System
- Structure Preparation
- Excitation of the structure

Introduction

- The measurements techniques used for modal testing are discussed:
  - Response measurement only
  - Force and response measurement
- The 2nd type of measurement techniques is of our concern:
  - Single-point excitation(SISO/SIMO)
  - Multi-point excitation (MIMO)

Test Planning

- Objective of the test
  - Levels according to Dynamic Testing Agency:

Level	Natural Freq	Damping ratio	Mode Shapes	Usabe for validation	Out of range residues	Updating
0						
1			Only in few points			
2						
3						
4						

Test Planning

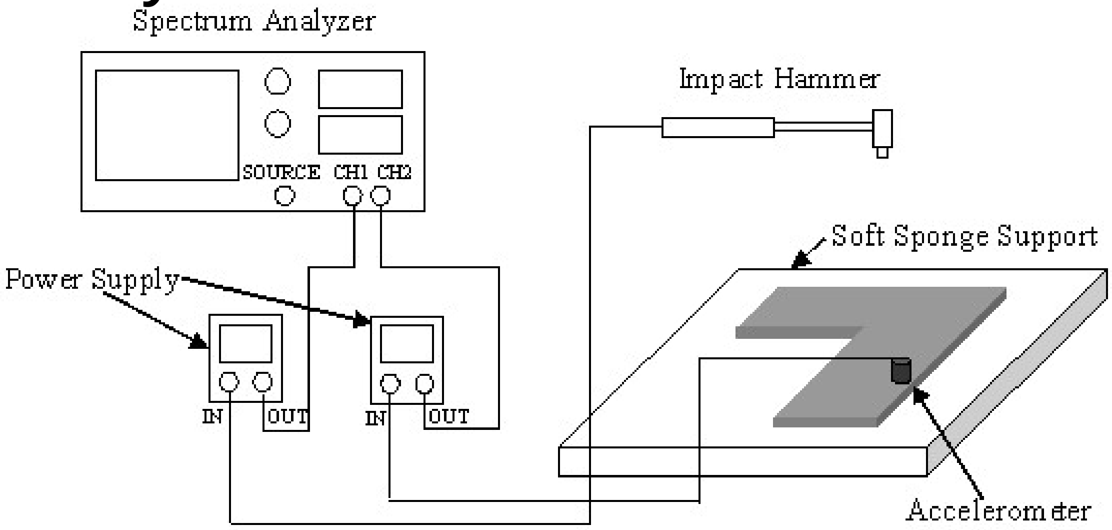
- Extensive test planning is required before full-scale measurement:
  - Method of excitation
  - Signal processing and data analysis
  - Proper selection of pickup points
  - Excitation location
  - Suspension method

Quality of measured data

- Signal quality
  - Sufficient strength and clarity/noise free
- Signal fidelity
  - No cross sensitivity
- Measurement repeatability
- Measurement reliability
- Measurement data consistency, including reciprocity

Basic Measurement System

- An excitation mechanism
- A transduction mechanism
- An Analyzer



Basic Measurement System

- Source of excitation signal:
  - Sinusoidal
  - Periodic (with specific freq. content)
  - Random
  - Transient
- Power Amplifier
- Exciter
- Transducers
- Condition Amplifiers
- Analyzers

Structure Preparation

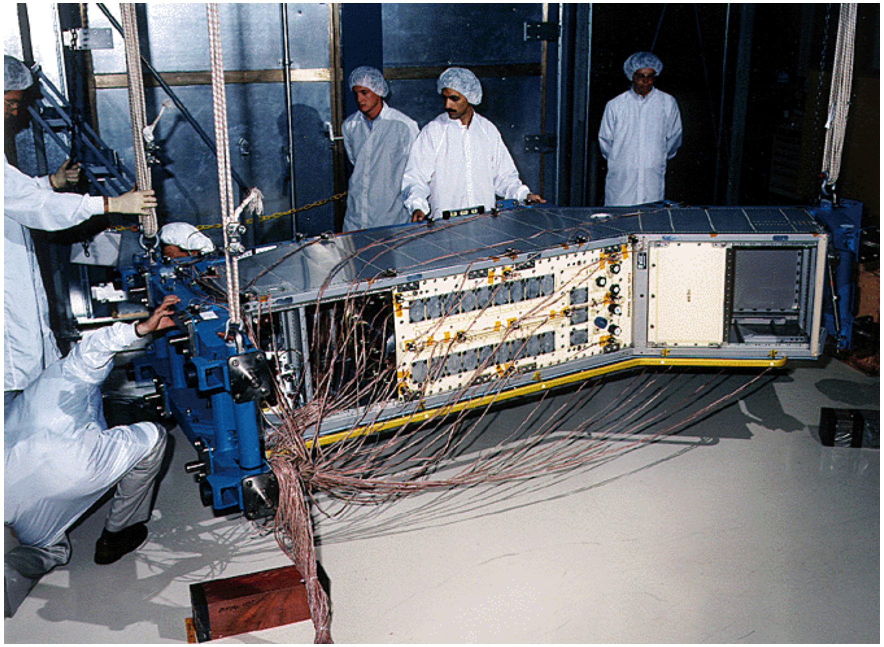
- Free Supports
- Grounded Support
- Loaded Support
- Perturbed Support

Free Supports

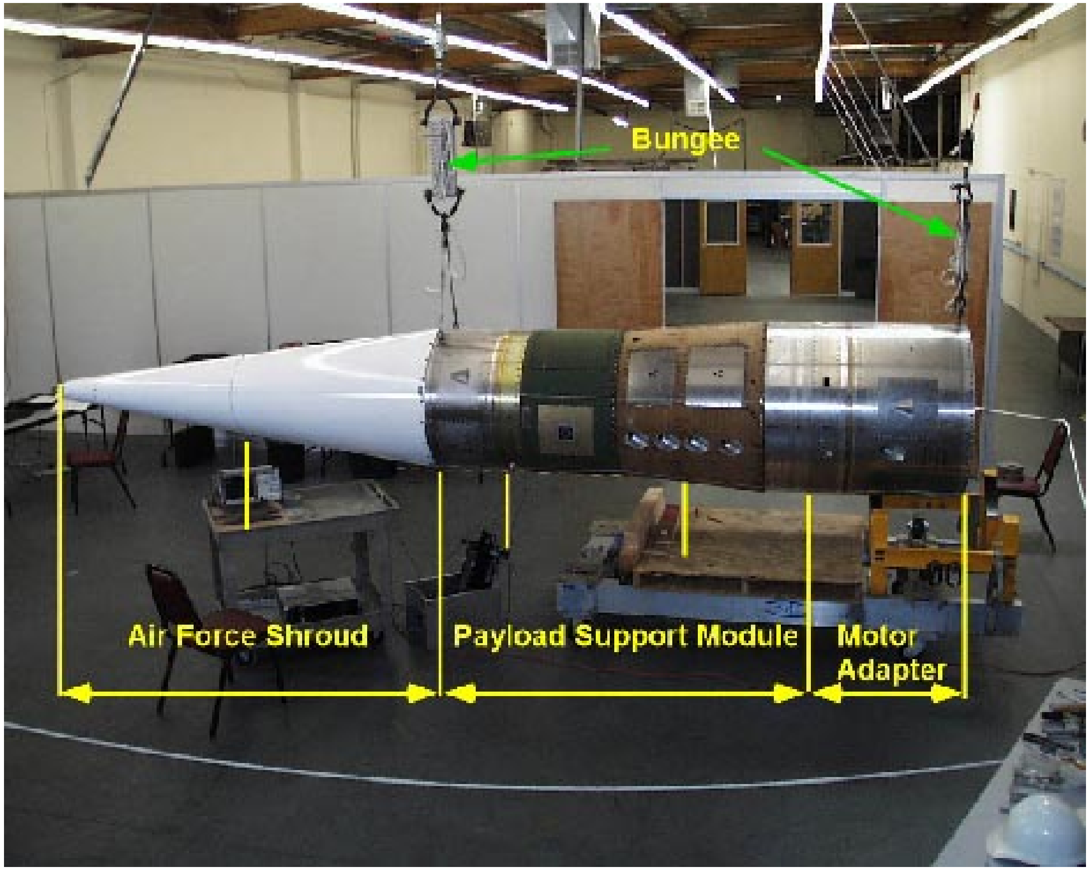
- Theoretically the structure will possess 6 rigid body modes @ 0 Hz.
- In practice this is provided by a soft support
- Rigid body modes are less then 10% of strain modes
  - Suspending from nodal points for minimum interference
  - The suspension adds significant damping to the lightly damped structures

Free Supports

- Suspension wires, should be normal to the primary vibration direction
- The mass and inertia properties can be determined from the RBMs.



Free Supports



Free Supports

Grounded Support

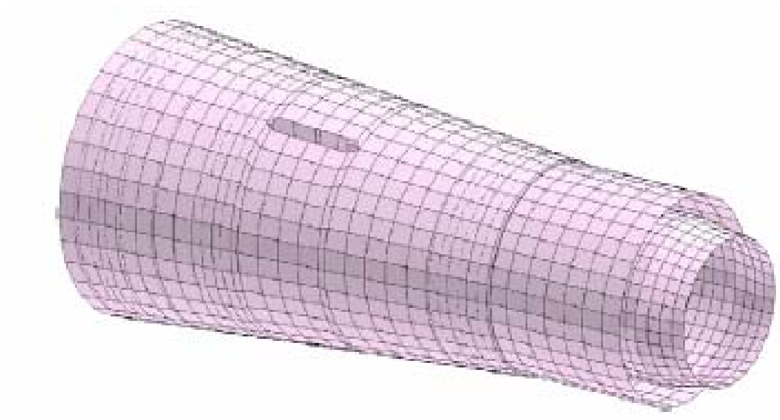
- The structure is fixed to the ground at selected points.
- The base must be sufficiently rigid to provide necessary grounding.
- Usually is employed for large structures
  - Parts of power generation station
  - Civil engineering structures
- Another application is simulating the operational condition
  - Turbine Blade
- Static stiffness can be obtained from low frequency mobility measurements.

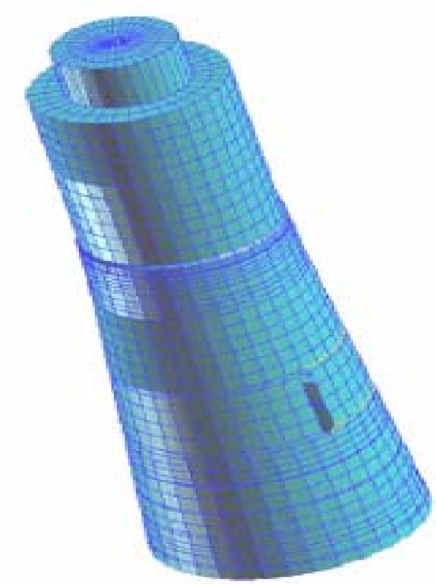
Loaded Support

- The structure is connected to a simple component with known mobility
  - A specific mass
- The effect of added mass can be removed analytically
- More modes are excited in a certain frequency range compared to free suspension
- The modes of structure are quite different

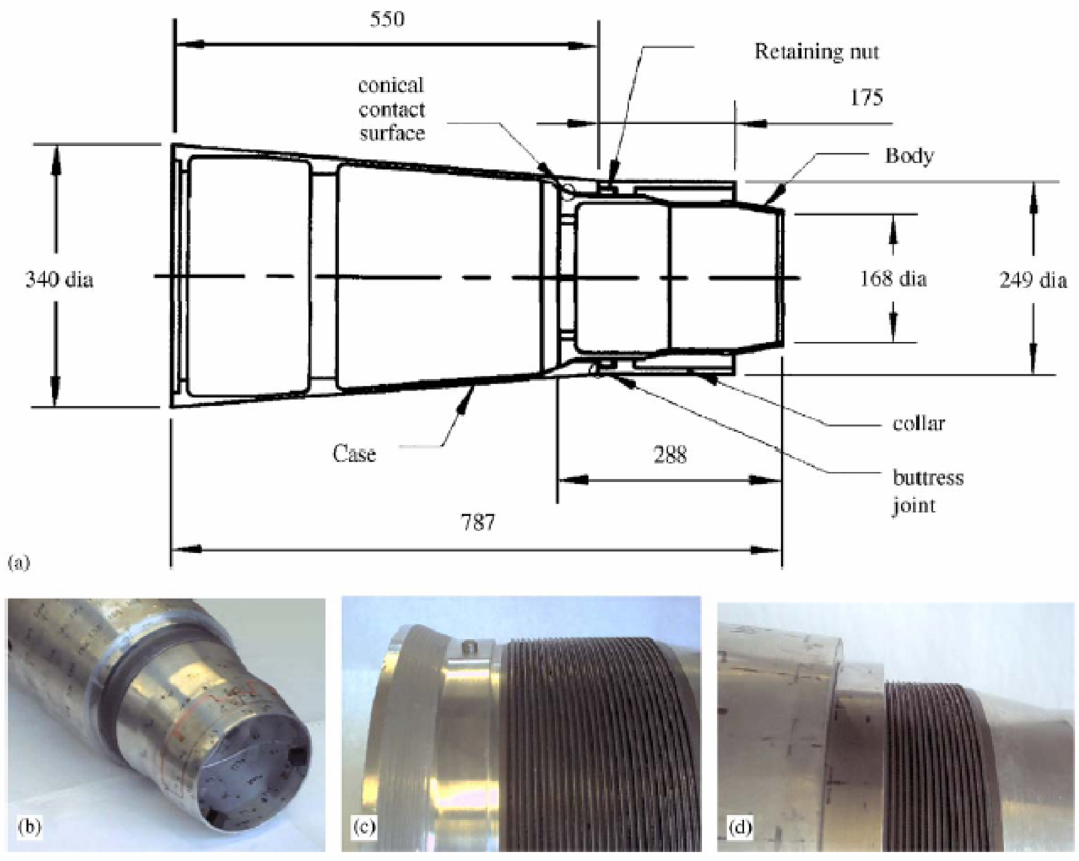
Perturbed Support

- The data base for the structure can be extended by repetition of modal tests for different boundary conditions

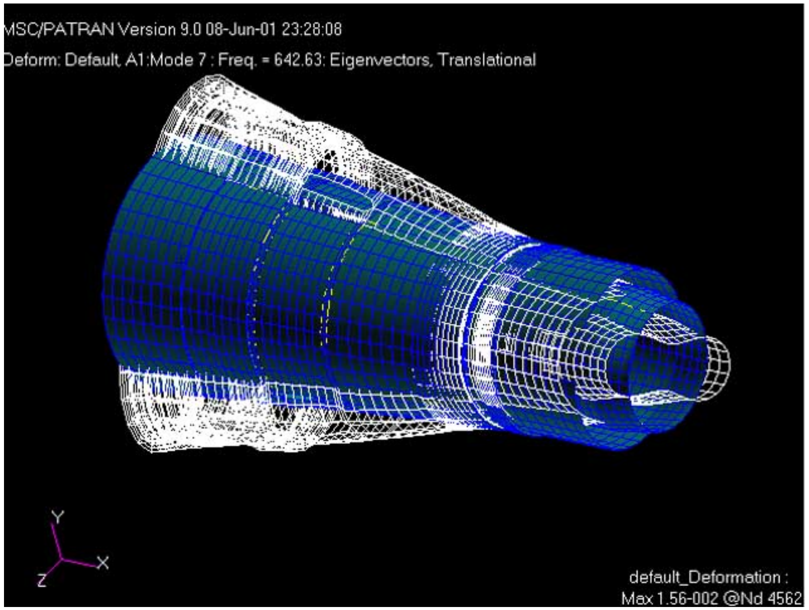


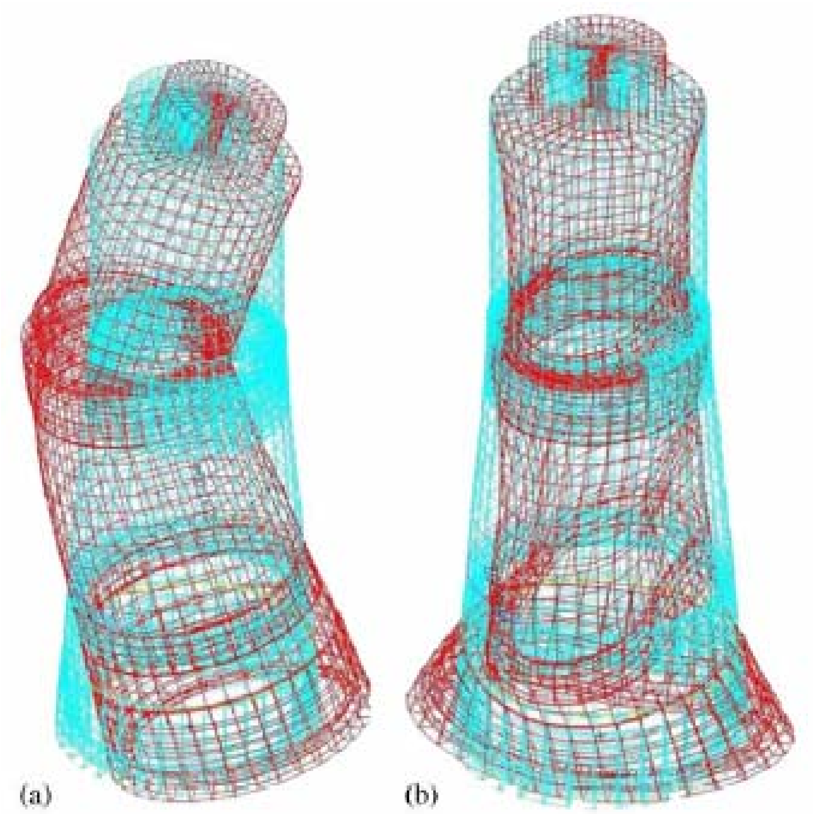


Perturbed Support



Perturbed Support





Excitation of the structure

- Various devices are available for exciting the structure:
  - Contacting
    - Mechanical (Out-of-balance rotating masses)
    - Electromagnetic (Moving coil in magnetic field)
    - Electrohydraulic
  - Non-Contacting
    - Magnetic excitation

Electromagnetic Exciters

- Supplied input to the shaker is converted to an alternating magnetic field acting on a moving coil.

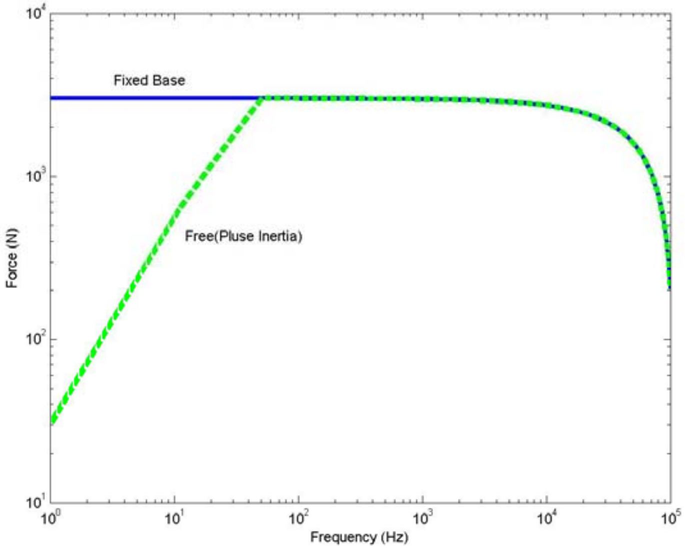
Electromagnetic Exciters

- There is a small difference between the force generated by the shaker and the applied force to the structure
  - The force required to accelerate the shaker moving
- The force required to excite the structure sharply reduces near the resonance point,
  - Much smaller than the generated force in the shaker and the inertia of the drive rod
  - Vulnerable to noise or distortion

Attachment to the structure

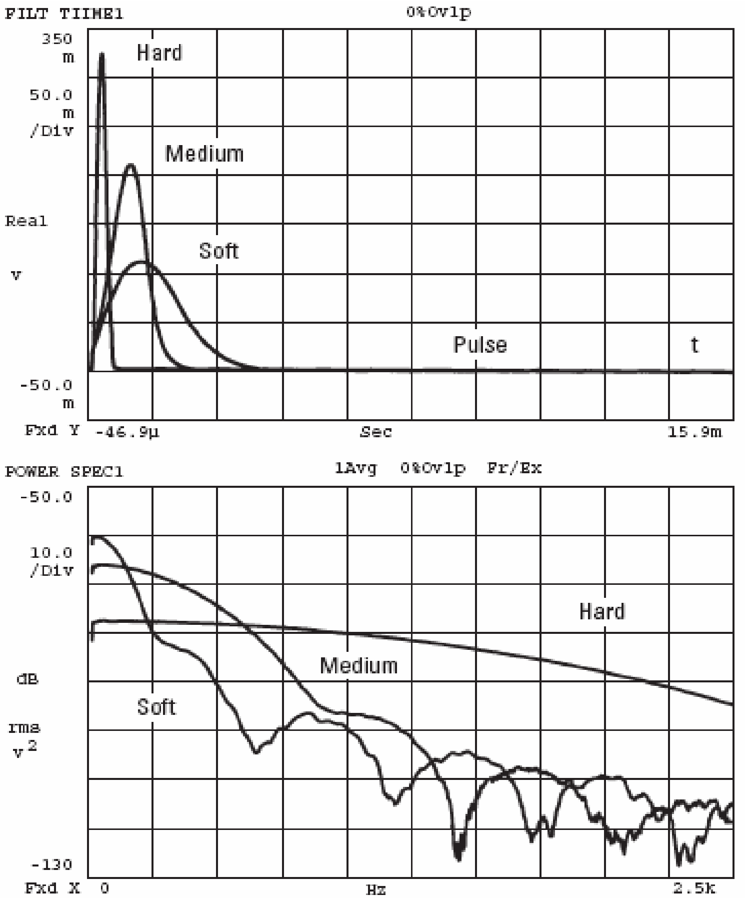
- Push rod or stingers:
  - Applying force in only one direction
  - Flexible drive rod/stinger introduces its own resonance into the measurement.

Support of shakers



Support of shakers

Hammer or Impactor Excitation



Other excitation methods

- Step Relaxation/sudden release
- Charge/Explosive impactor
- ….

Moving Support

- Corresponds to grounded model
- Only responses are measured
- When the mass properties are known, the modal properties can be calculated from measured data

Moving Support

Modal Testing

(Lecture 12)

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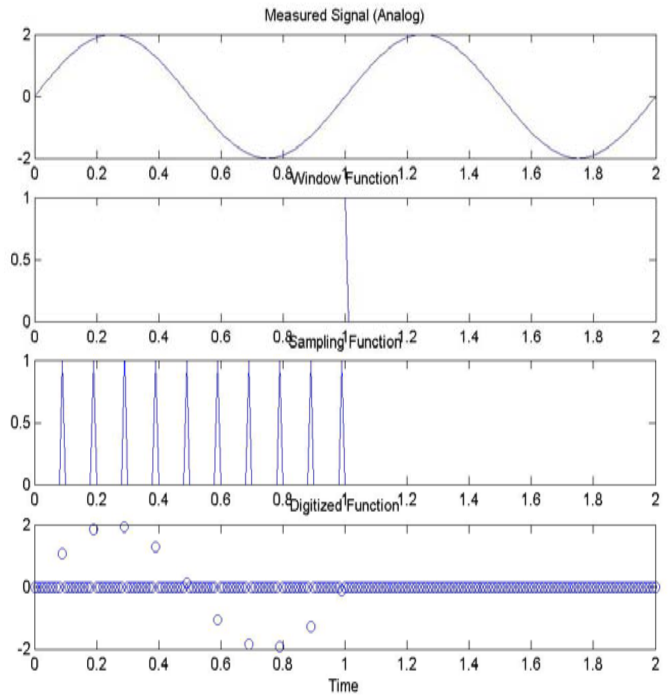
Digital Signal Processing

- Introduction
- Basics of Discrete Fourier Transform (DFT)
- Aliasing
- Leakage
- Windowing
- Filtering
- Improving Resolution

Introduction

- The measured force or accelerometer signals are in time domain.
- The signals are digitized by an A/D converter
- And recorded as a set of N discrete values evenly spaced in the period T



Basics of DFT

- The spectral properties of the recorded signal can be obtained using Discrete Fourier Transform/Series (DFT/DFS):
  - The DFT assumes the signal x(t) is periodic
  - In the DFT there are just a discrete number of items of data in either form
    - There are just N values x_k
    - The Fourier Series is described by just N values

Basics of DFT

$$x(t) = x(t+T)$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

$$\omega_n = \frac{2\pi n}{T},$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(\omega_n t)dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(\omega_n t)dt$$

or

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{i\omega_n t}$$

$$X_n = \frac{1}{T}\int_0^T x(t)e^{-i\omega_n t}dt$$

$$X_{-n} = X_n^*$$

$$x_k = \frac{a_0}{2} + \sum_{n=1}^{\frac{N-1}{2}} a_n \cos(\frac{2\pi nk}{N}) + b_n \sin(\frac{2\pi nk}{N})$$

$$a_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \cos(\frac{2\pi nk}{N})$$

$$b_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \sin(\frac{2\pi nk}{N})$$

or

$$x_k = \sum_{n=0}^{N-1} X_n e^{2\pi ink/N}$$

$$X_n = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-2\pi ink/N}, X_{N-r} = X_r^*$$

Basics of DFT

$$X_n = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-2\pi i n k / N}$$

$$\begin{Bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{Bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \uparrow & \nearrow & \cdots & \nwarrow \\ \vdots & \cdots & \cdots & \vdots \\ \uparrow & \cdots & \cdots & \nearrow \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{Bmatrix}$$

Basics of DFT

- The sampling frequency:

$$f_s = \frac{1}{t_s} = \frac{N}{T} \Rightarrow \omega_s = \frac{2\pi}{t_s} = \frac{2\pi N}{T}$$

Nyquist Frequency

- The range of frequency spectrum:

$$f_{\max} = \frac{f_s}{2} = \Rightarrow \omega_{\max} = \frac{\omega_s}{2} = \frac{\pi N}{T}$$

- The resolution of frequency spectrum:

$$\Delta f = \frac{1}{T}, \Delta\omega = \frac{2\pi}{T}$$

Basics of DFT

- There are a number of features of DF analysis which if not properly treated, can give rise to erroneous results:
  - Aliasing
    - Mis-interoperating a high frequency component as a low frequency one
  - Leakage
    - Periodicity of the signal

Aliasing

- Digitizing a ‘low’ frequency signal produces exactly the same set of discrete values as result from the same process applied to a higher frequency signal

$$\omega < \frac{\omega_s}{2}$$

$$\omega_s - \omega$$

Aliasing

$$Compare:$$

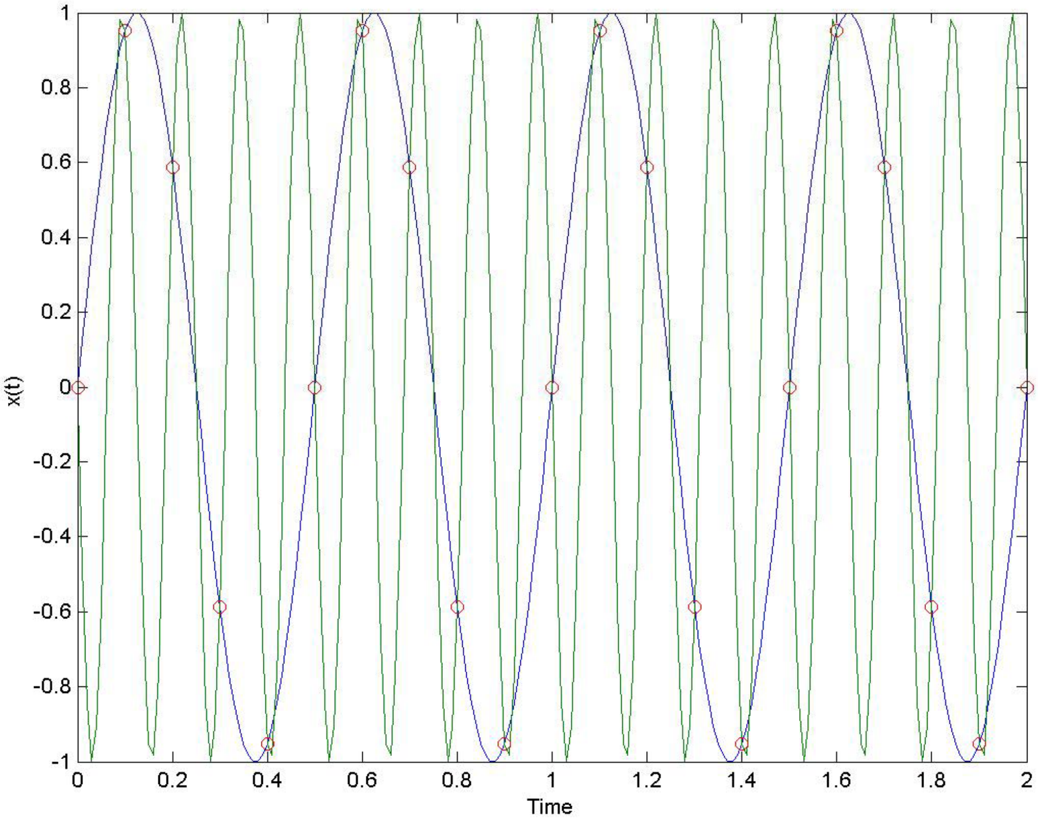
$$\sin(2\pi p\frac{k}{N}) \Leftrightarrow \quad \sin(2\pi(N-p)\frac{k}{N})$$

$$\sin(2\pi k - \frac{2\pi pk}{N})$$

$$-\sin(\frac{2\pi pk}{N})$$

$$p < N/2$$

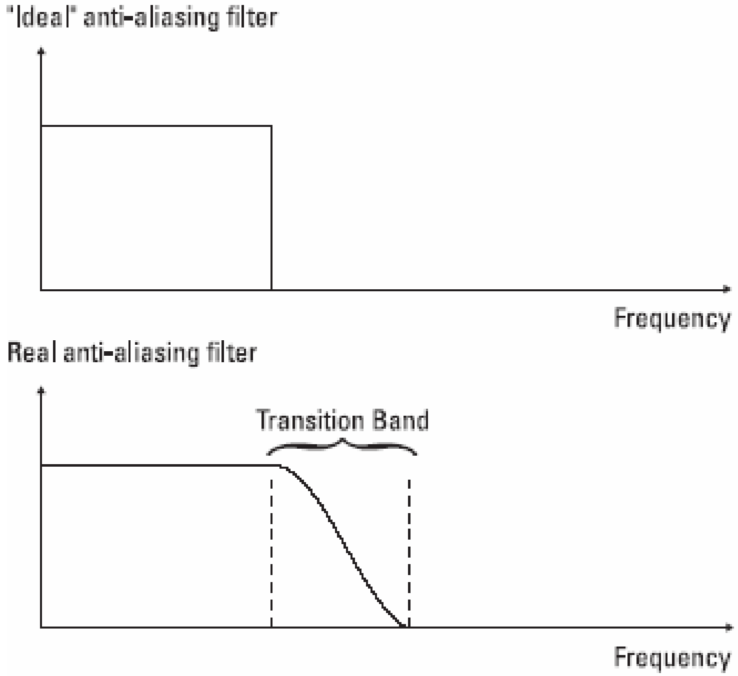
Aliasing



Aliasing

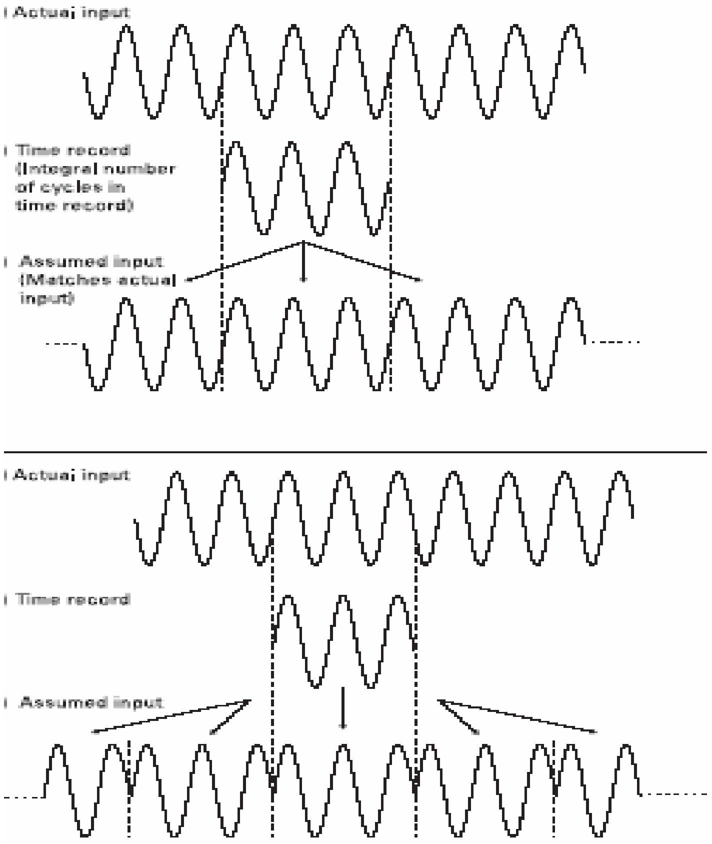
- The solution to the problem is to use an anti-aliasing filter
  - Subjecting the original signal to low pass with sharp filter
  - Filters have a finite cut-off rate; it is necessary to reject the spectral range near Nayquist frequency

$$\omega > (08 - 1.0)\frac{\omega_s}{2}$$

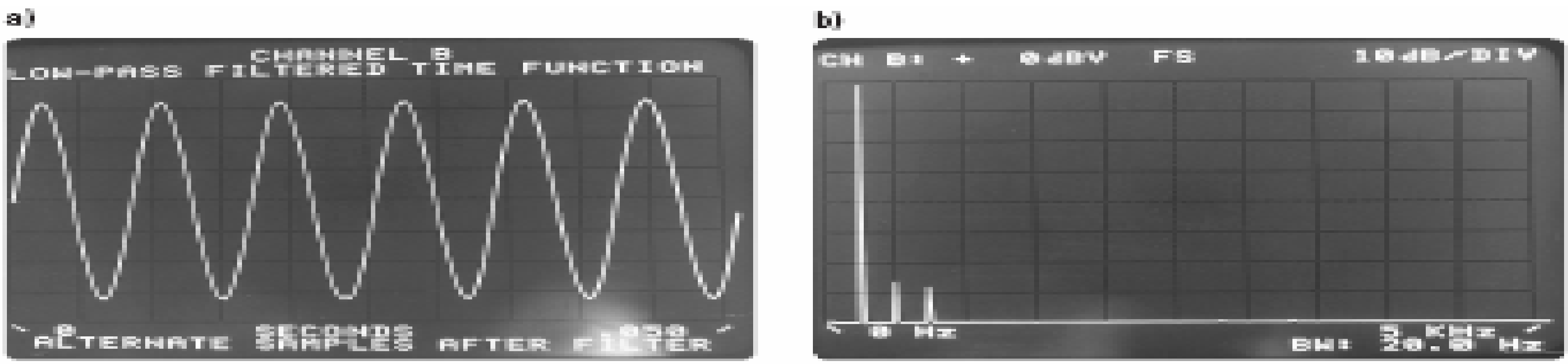


Leakage

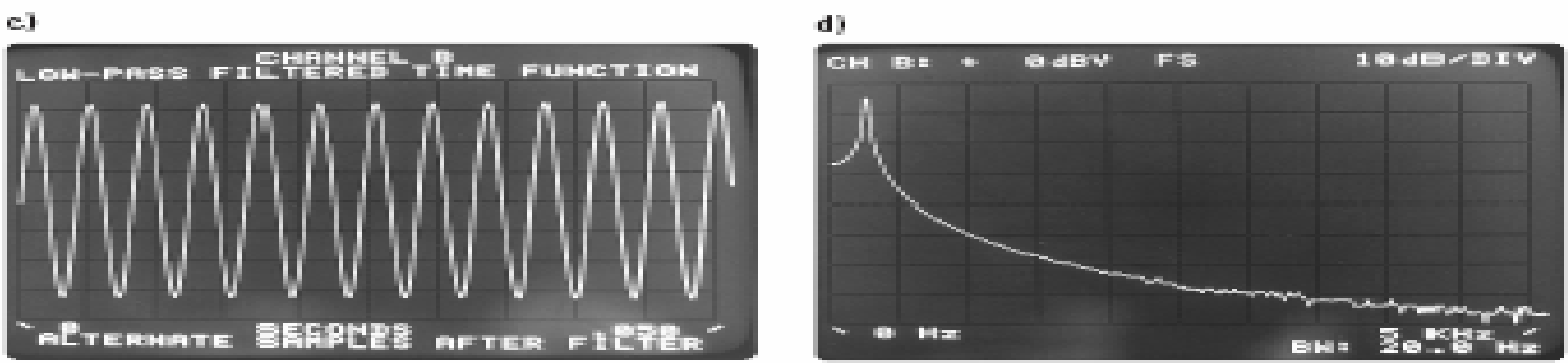
- A direct consequence of taking a finite length of time history coupled with assumption of periodicity
- Energy is leaked into a number of spectral lines close to the true frequency.



Leakage



a) & b) Sine wave periodic in time record



c) & d) *Sine wave not periodic in time record*

Leakage

- To avoid the leakage there are number of scenarios:
  - Increasing the record time T
  - Windowing
    - Multiply the time record by a function that is zero at the ends of the time record and large in the middle, the FFT content is concentrated on the middle of the time record

Windowing

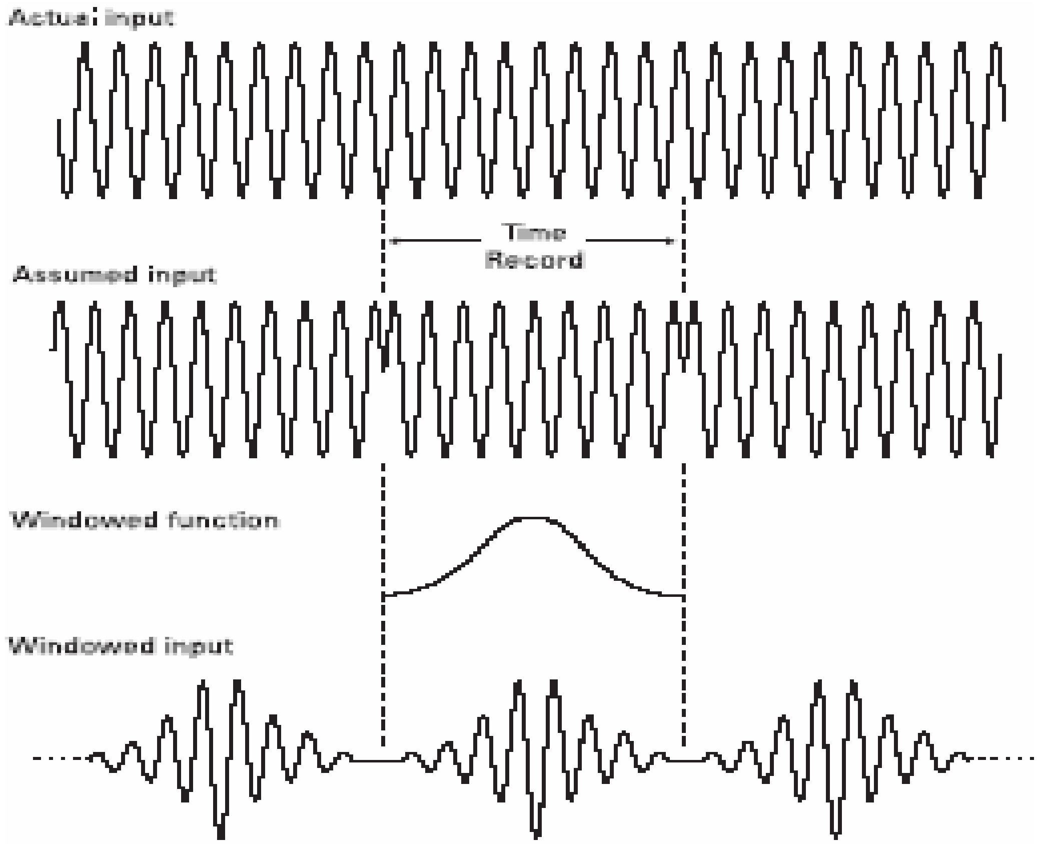
- Windowing involves the imposition of a prescribed profile on the time signal prior to performing the FT

$$x'(t) = w(t) \times x(t)$$

$$w(t) = \begin{cases} a_0 - a_1 \cos(\omega_0 t) + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) & \\ -a_3 \cos(3\omega_0 t) + a_4 \cos(4\omega_0 t) & 0 < t < T, \\ 0 & elsewhere \end{cases}$$

$$\omega_0 = {}^{2\pi}\!/_{T}$$

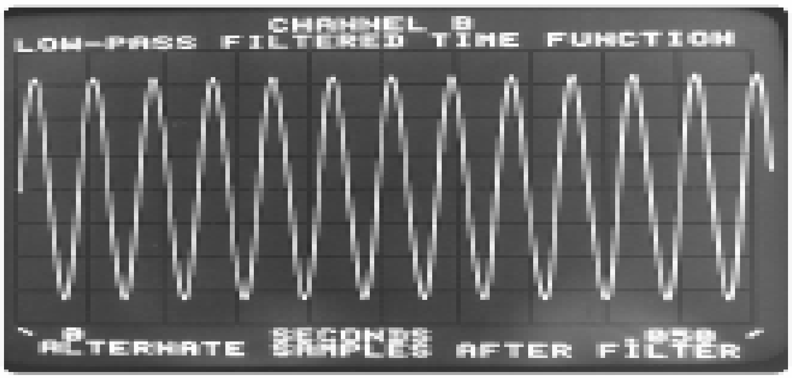
Windowing



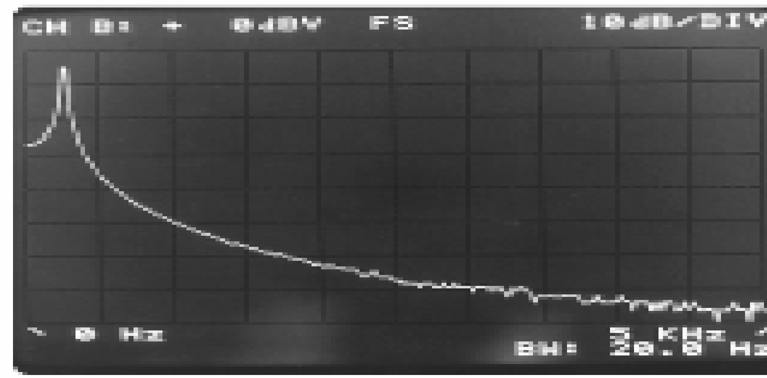
Actual input
Time
Record
Assumed input
Windowed function
Windowed input

Windowing

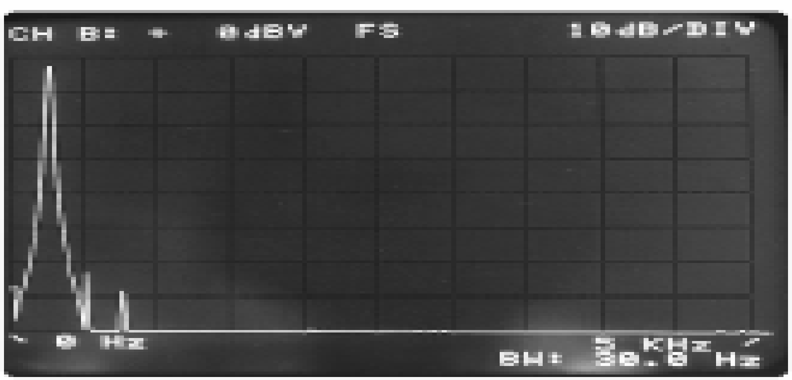
a) Sine wave not periodic in time record



b) FFT results with no window function



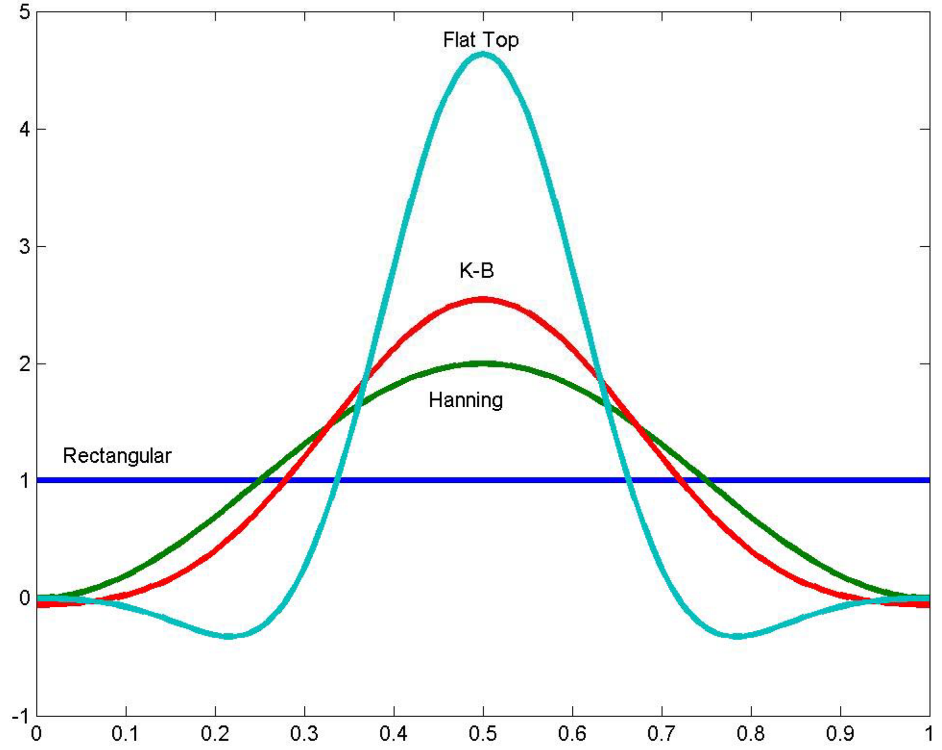
c) FFT results with a window function



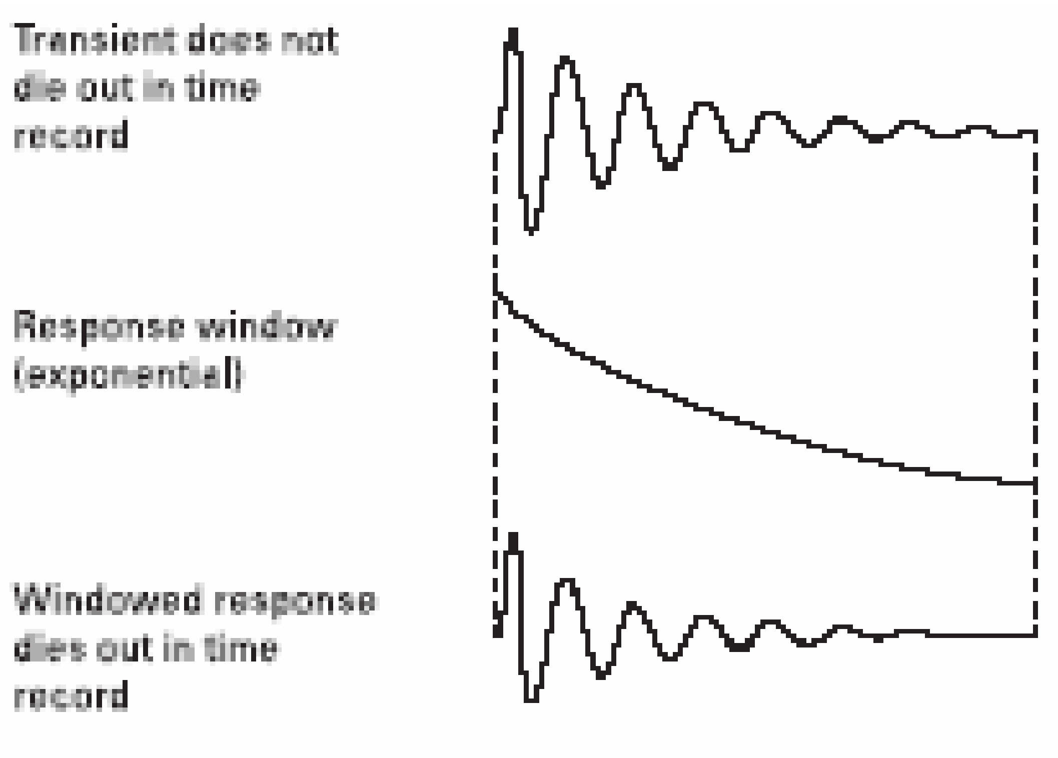
Windowing

Function	a_0	a_1	a_2	a_3	a_4
Rectangular	1	-	-	-	-
Hanning	1	1	-	-	-
Kaser-Bessel	1	1.298	0.244	0.003	-
Flat top	1	1.933	1.286	0.388	0.032

Windowing


Flat Top
K-B
Hanning
Rectangular
5
4
3
2
1
0
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Windowing



Transient does not die out in time record
Response window (exponential)
Windowed response dies out in time record

Improving Resolution (Zoom)

- There arises limitations of inadequate frequency resolution
  - at the lower end of the frequency range
  - For lightly-damped systems
- A common solution is to concentrate all spectral lines into a narrow band
  - Within f_{min}-f_{max}
  - Instead of 0-f_{max}

Zoom

- Method 1:
  - Shifting the frequency origin of the spectrum

$$x(t) = A\sin(\omega t)$$

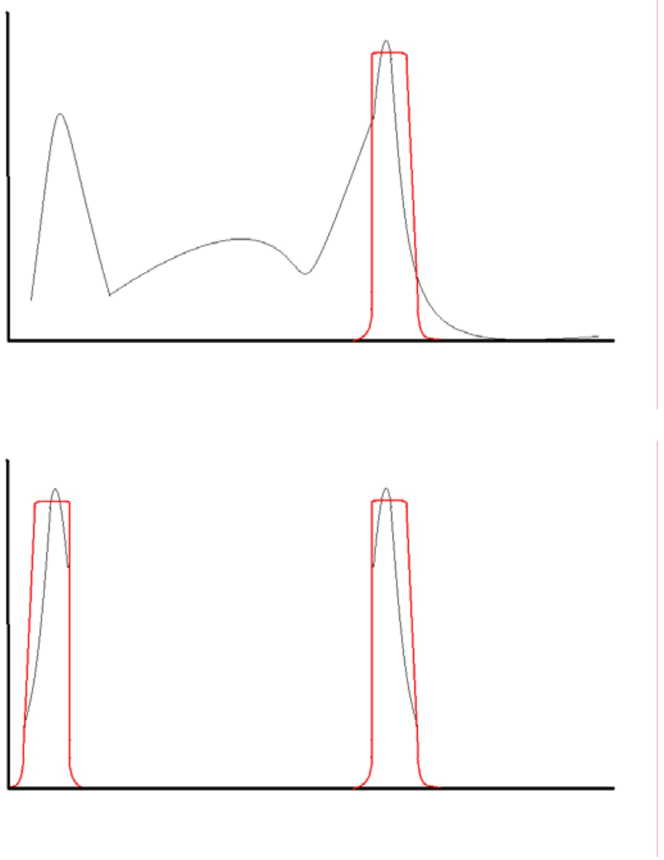
$$x'(t) = A\sin(\omega t) \times \cos(\omega_{\min} t)$$

$$\Rightarrow \frac{A}{2}\left[\sin(\omega - \omega_{\min})t + \sin(\omega + \omega_{\min})t\right]$$

- The modified signal is then analysed in the range of 0-(f_{max}-f_{min})

Zoom

- Method 2:
  - A controlled aliasing effect
    - Applying a band pass filter
    - Because of the aliasing phenomenon, the frequency component between f_1 and f_2 will appear aliased between 0-(f_2-f_1)



Modal Testing

(Lecture 13)

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Use of Different Excitation Signals

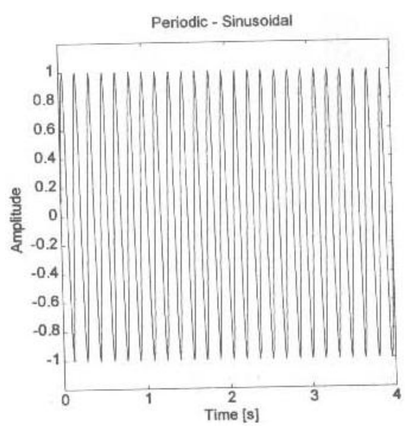
- Introduction
- Stepped-Sine Testing
- Slow Sine Sweep Testing
- Periodic Excitation
- Random Excitation
- Transient Excitation

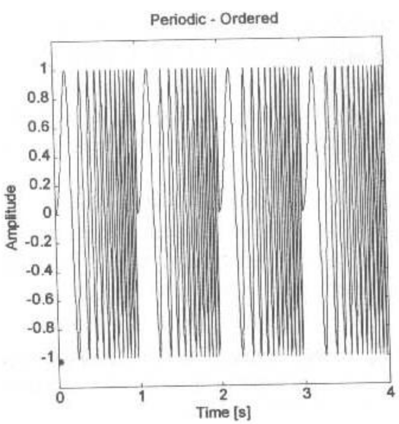
Introduction

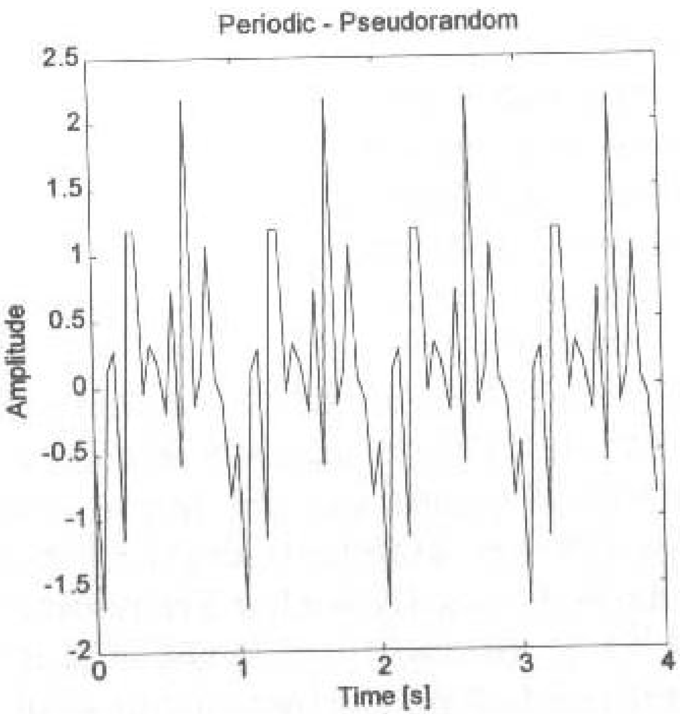
- There are three different classes of excitation signals used:
  - Periodic
  - Transient
  - Random

Introduction

- Periodic:
  - Stepped sine
  - Slow sine sweep
  - Periodic
  - Pseudo-random

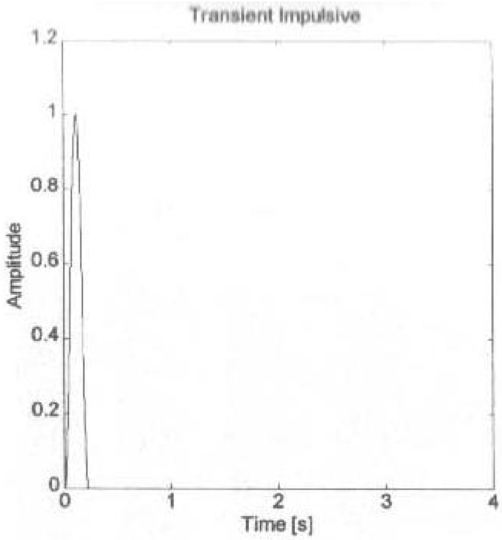


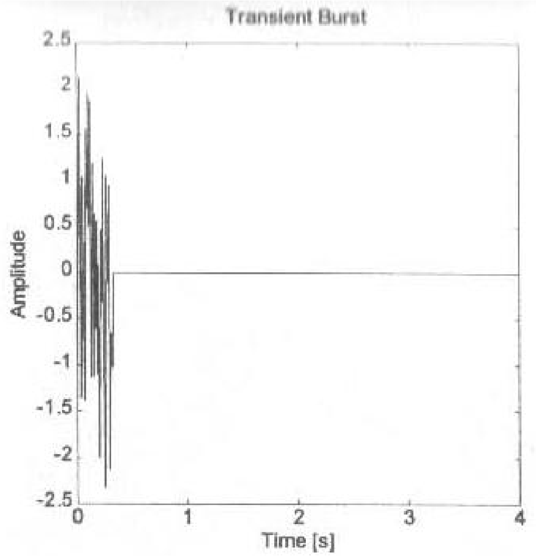




Introduction

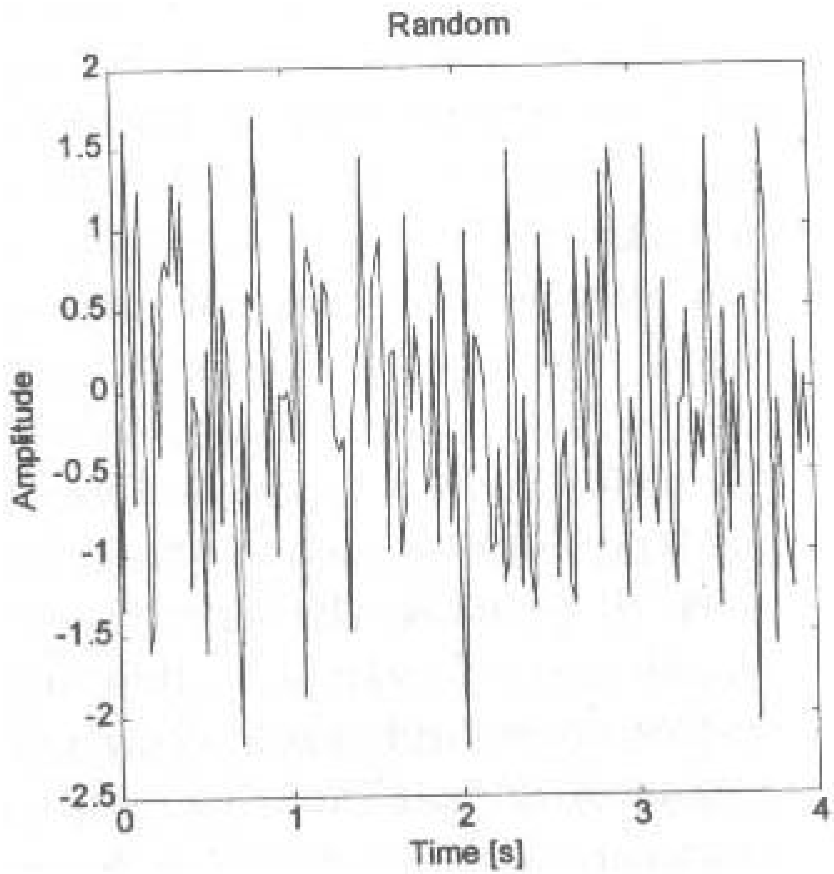
- Transient:
  - Burst sine
  - Burst random
  - Chirp
  - Impulse





Introduction

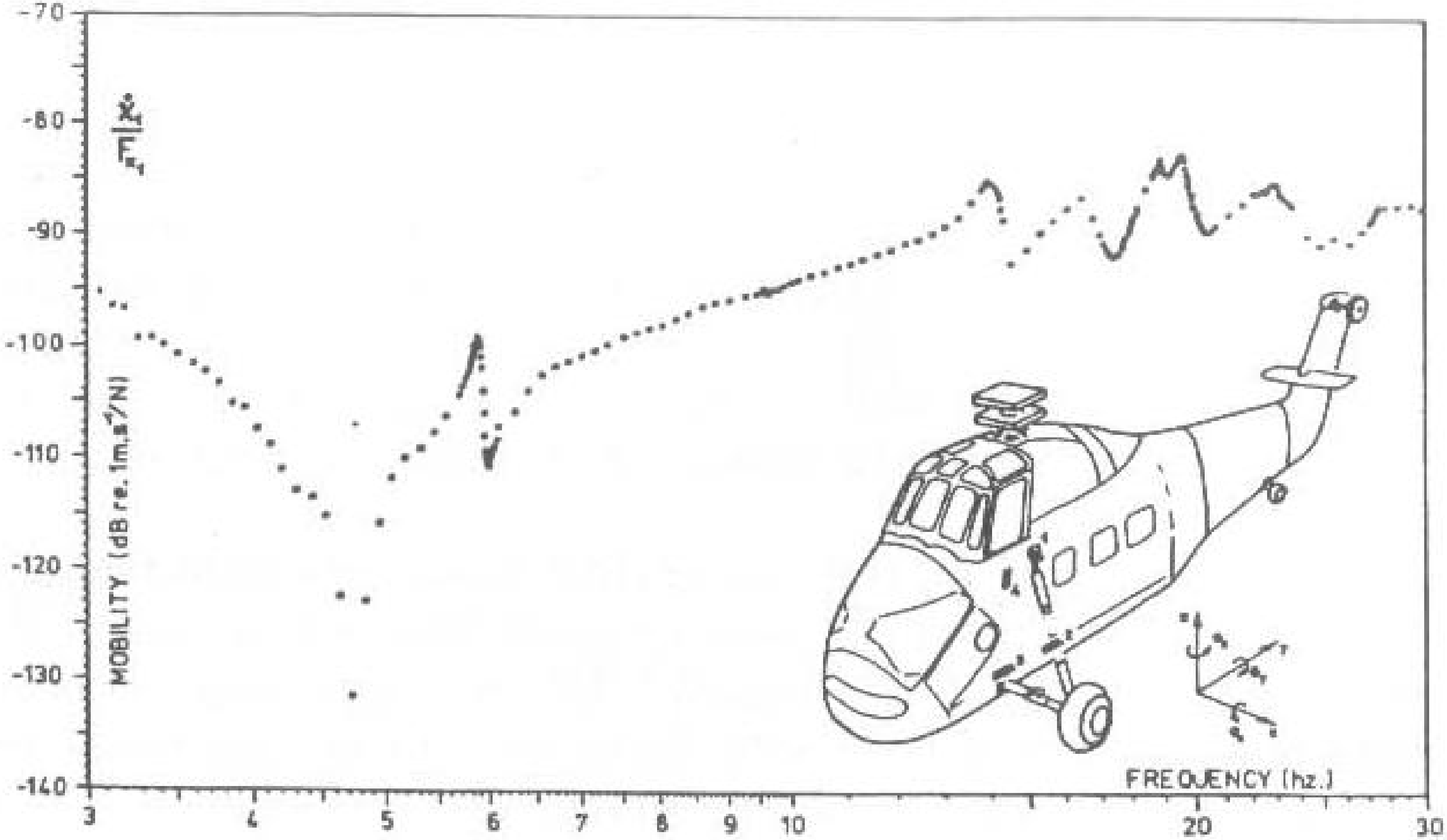
- Random:
  - (true) random
  - White noise



Stepped-Sine Testing

- Classical method of FRF measurement
- To encompass a frequency range of interest, the command signal frequency is stepped from one frequency to another
  - The excitation/response(s) are measured (amplitudes and phase(s)) .
  - It is necessary to ensure that the steady-state condition is attained before the measurement.

Stepped-Sine Testing



Stepped-Sine Testing

- The extent of unwanted transient response depends on:
  - Proximity of excitation frequency to a natural frequency,
  - The abruptness of the changeover from the previous command signal to the new one,
  - The lightness of the damping of nearby modes.

Stepped-Sine Testing

- An advantage of stepped-sine testing is the facility of taking measurement where and as they are required.

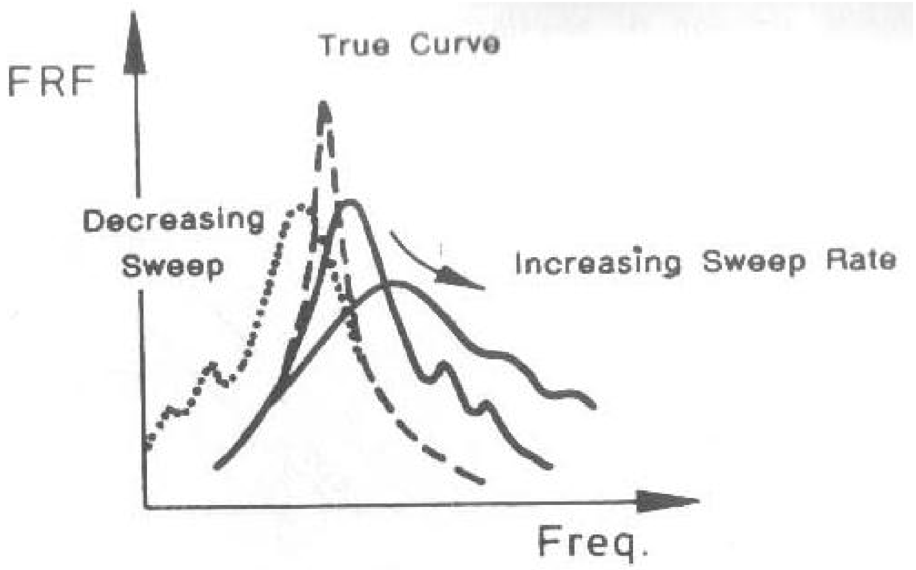
No. point between HPP's	Largest Error	
	%	dB
1	30	3
2	10	1
3	5	0.5
5	2	0.2
8	1	0.1

Slow Sine Sweep Testing

- Involves the use of a sweep oscillator
  - Provides a sinusoidal signal
  - Its frequency is varied slowly but continuously
- If an excessive sweep rate is used then distortions of FRF plot are introduced

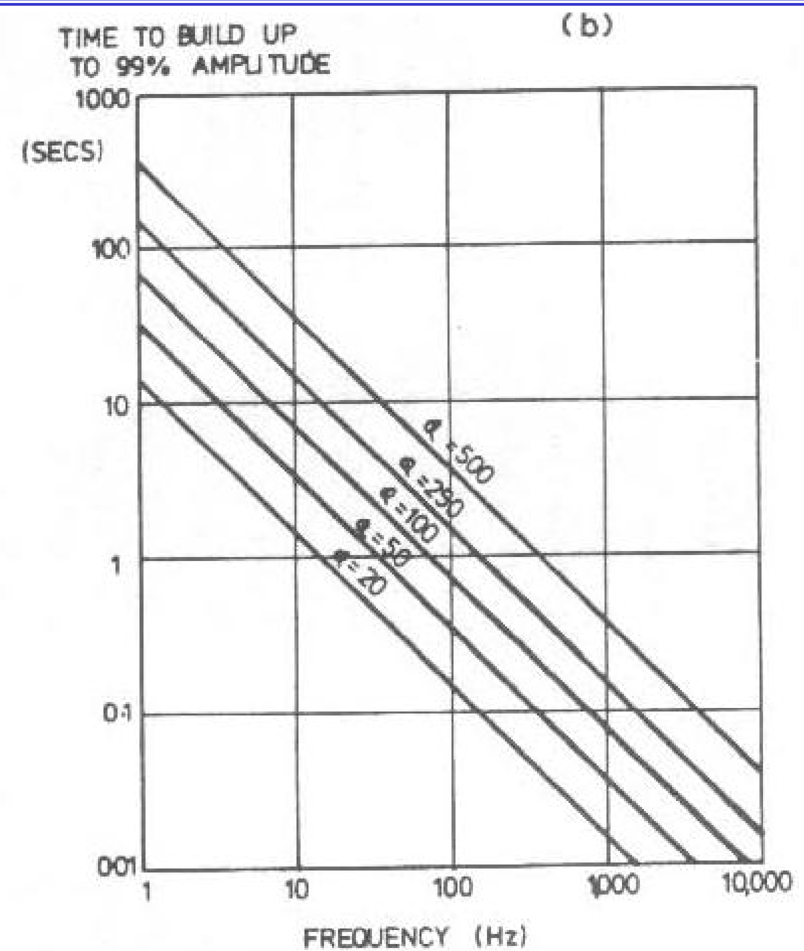
Slow Sine Sweep Testing

- One way of checking the suitability of a sweep rate is to make the measurement twice:
  - Once sweeping up
  - And the 2nd time sweeping down



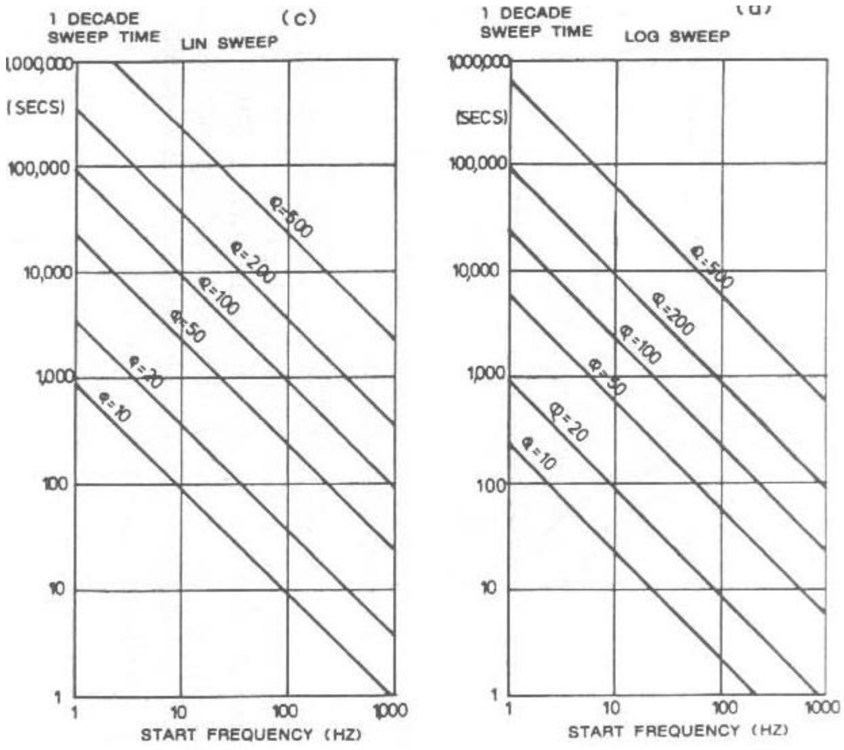
Slow Sine Sweep Testing

- It is possible to prescribe an optimum sweep rate for a given structure taking into account its damping levels



Slow Sine Sweep Testing

- Recommended sweep rate:



Slow Sine Sweep Testing

- ISO prescribes maximum linear and log sweep rate through a resonance as:

$$Linear$$

$$S_{\max} < 216 \times (\zeta_r \omega_r)^2 \, Hz / \min$$

$$Log$$

$$S_{\max} < 310 \times (\zeta_r^2 \omega_r) \qquad Octaves / \min$$

Periodic Excitation

- A natural extension of the sine wave test methods:
  - To use a complex periodic input signal which contains all the frequencies of interest,
  - The DFT of both input and output signals are computed and the ratio of these gives the FRF
  - Both signal have the same frequency contents

Periodic Excitation

- Two types of periodic signals are used:
  - A deterministic signal (square wave)
    - Some frequency components are inevitably weak.
  - Pseudo-Random type of signal
    - The frequency components may be adjusted to suit a particular requirements-such as equal energy at each frequency,
    - Its period is exactly equal to the sampling time resulting zero leakage .

Random Excitation

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{fx}(\omega) = H(\omega) S_{ff}(\omega)$$

$$S_{xx}(\omega) = H(\omega) S_{xf}(\omega)$$

$$H_1(\omega) = \frac{S_{fx}(\omega)}{S_{ff}(\omega)}$$

$$H_2(\omega) = \frac{S_{xx}(\omega)}{S_{xf}(\omega)}$$

$$\gamma^2 = \frac{H_1(\omega)}{H_2(\omega)}$$

Random Excitation

- There may be noise on one of the two signals
  - Near resonance this is likely to influence the force signal
  - At anti-resonances it is the response signal which will suffer

Random Excitation

- H_2 might be a better indication near resonances while H_1 is a better indication near anti-resonances:

$$H_1(\omega) = \frac{S_{fx}(\omega)}{S_{ff}(\omega) + S_{nn}(\omega)}, \quad H_2(\omega) = \frac{S_{xx}(\omega) + S_{mm}(\omega)}{S_{xf}(\omega)}$$

Auto-spectra of noise on the output signal

Auto-spectra of noise on the input signal

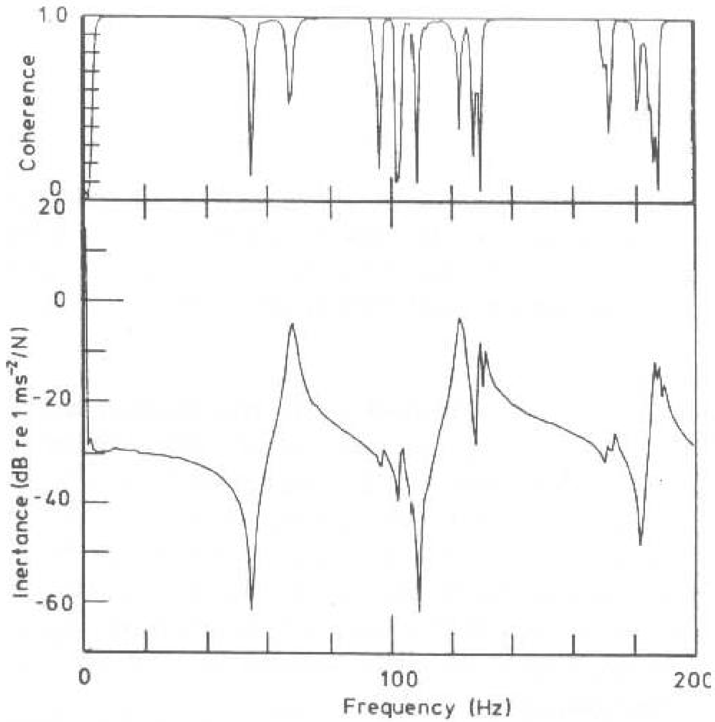
Random Excitation

- A closer optimum formula for the FRF is defined as the geometric mean of the two standard estimates
  - Phase is identical to that in the two basic estimates

$$H_v(\omega) = \sqrt{H_1(\omega) H_2(\omega)}$$

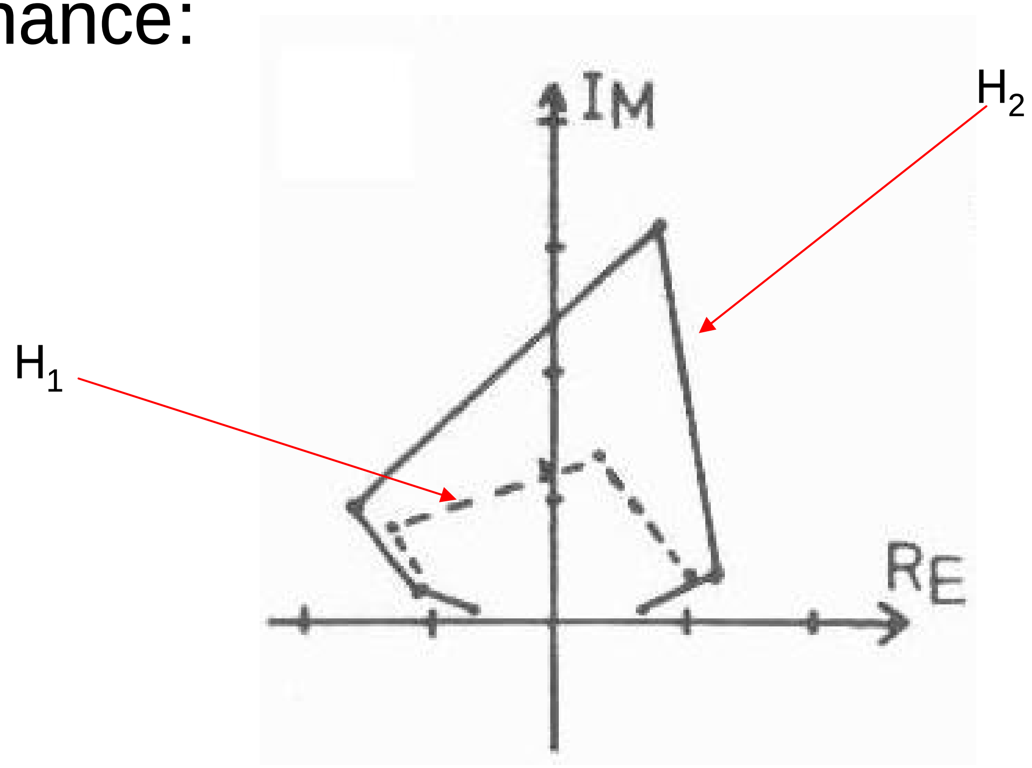
Random Excitation

- Typical measurement made using random excitation:



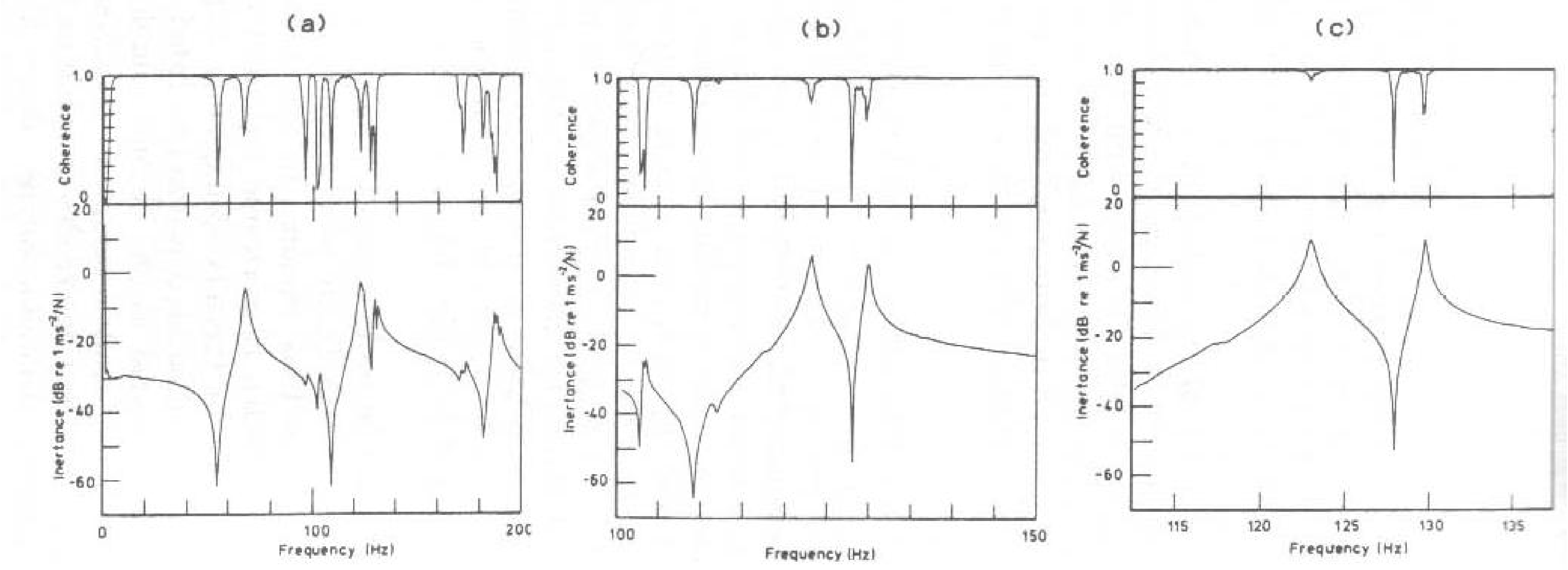
Random Excitation

- Details from previous plot around a resonance:



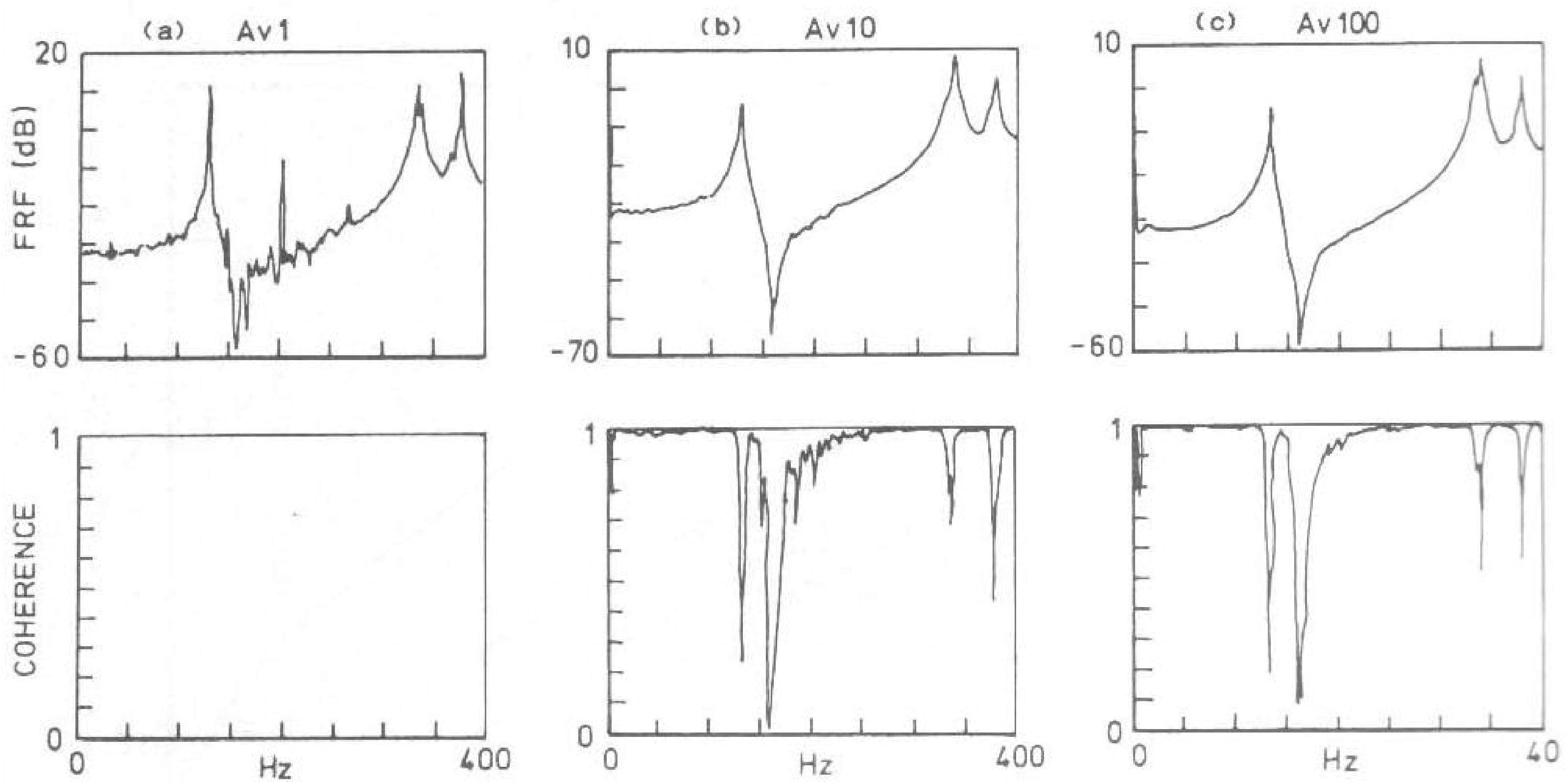
Random Excitation

- Use of zoom spectrum analysis:
  - Improving the resolution removes the major source of low coherence



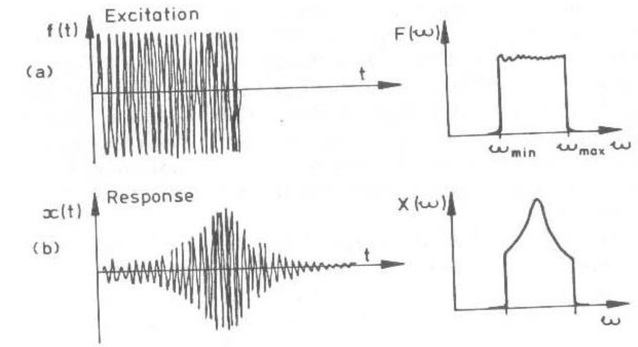
Random Excitation

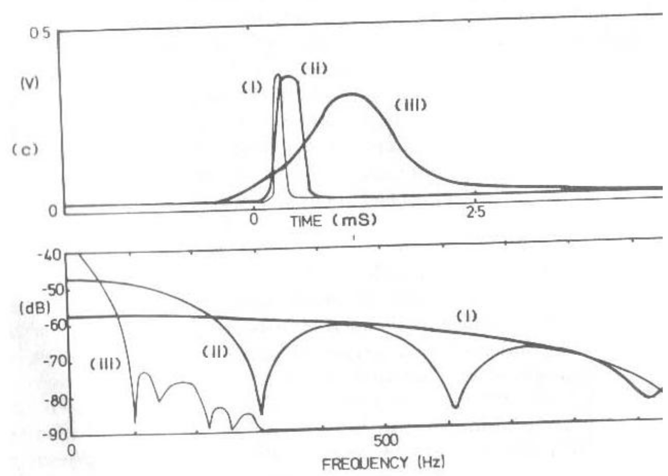
- Effect of averaging:



Transient Excitation

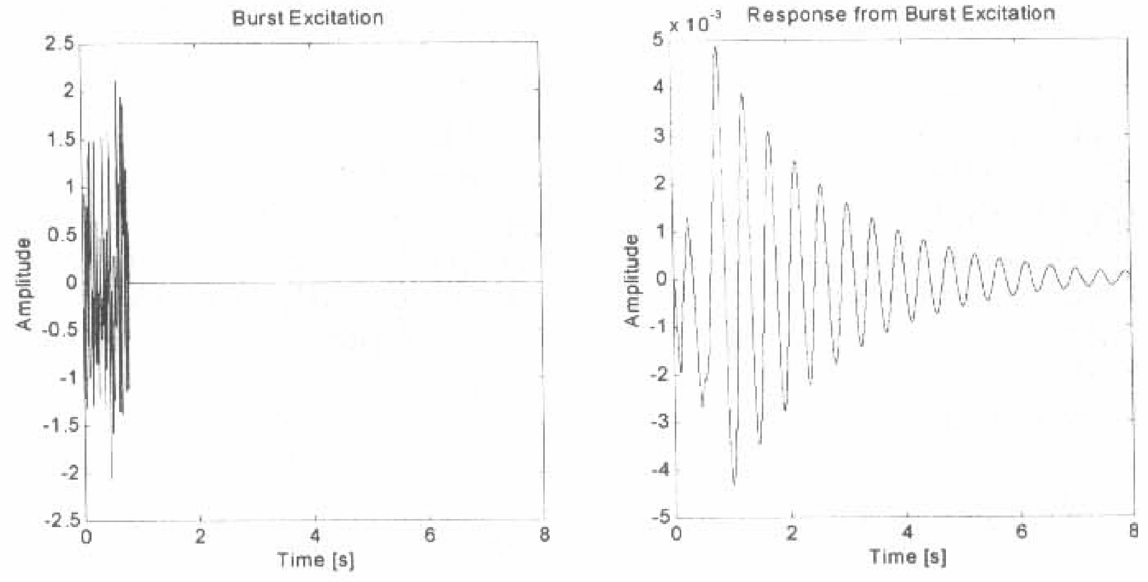
- The excitation and the response are contained within the single measurement





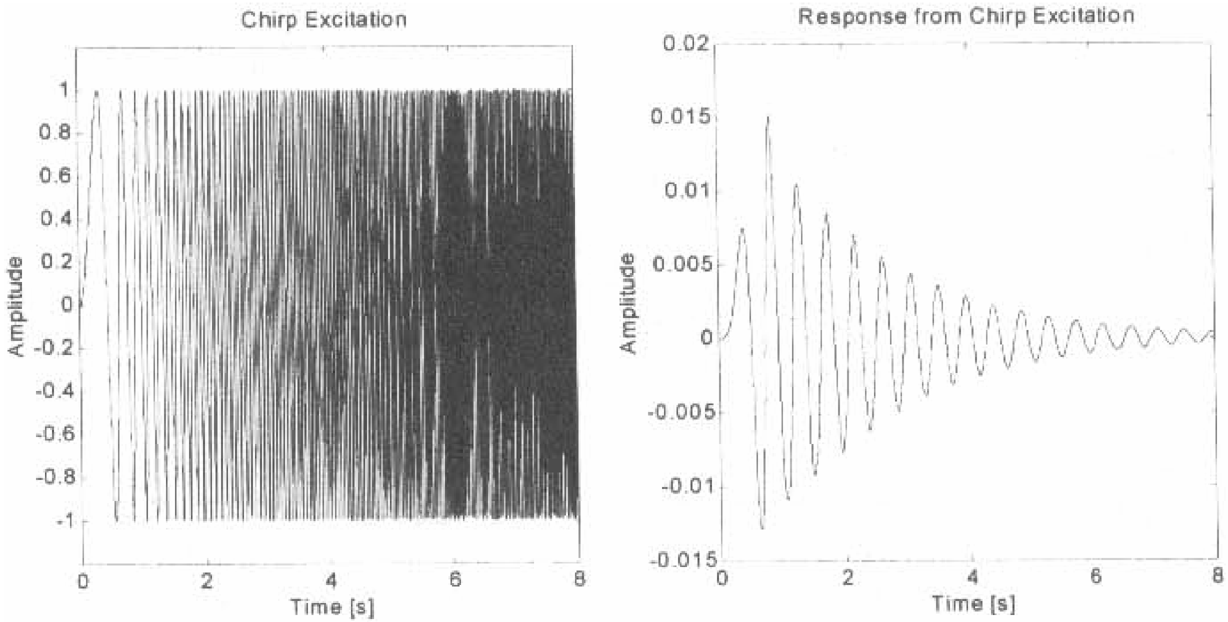
Transient Excitation

- Burst excitation signals:
  - A short section of a continuous signal (sin, random, …) followed by a period of zero wave.



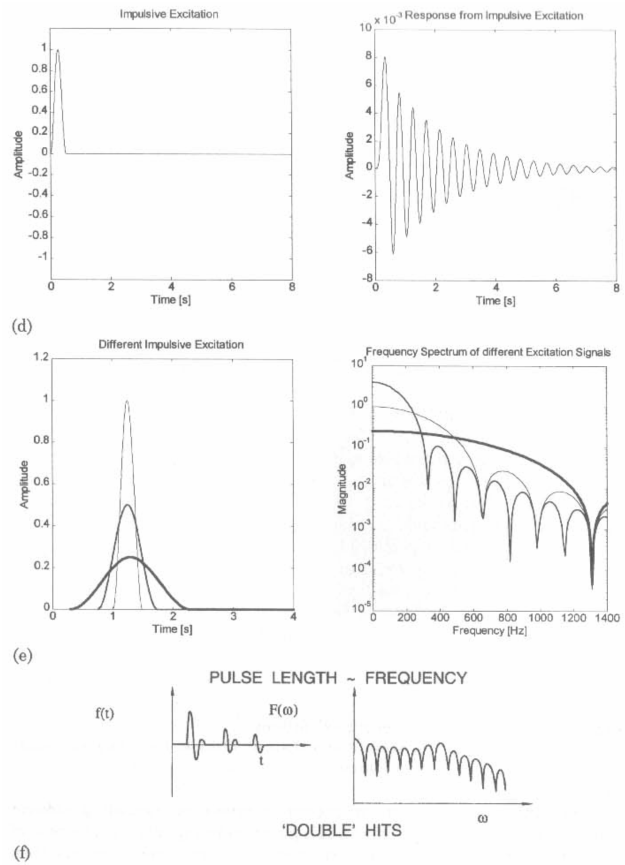
Transient Excitation

- Chirp excitation:
  - The spectrum can be strictly controlled to be such within frequency range of interest



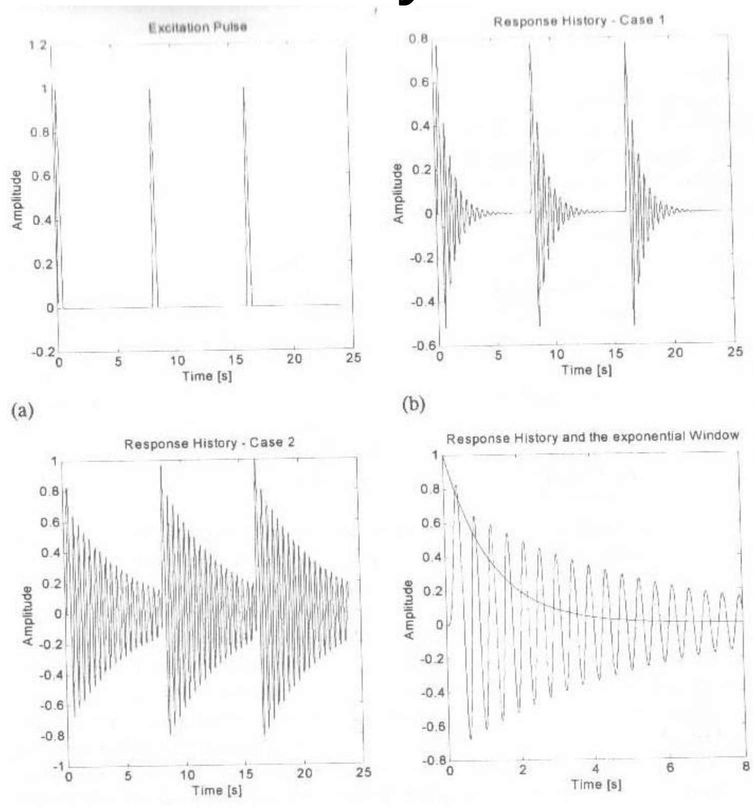
Transient Excitation

- Impulsive excitation by Hammer:
  - Different impulsive excitations
  - Signals and spectra for double hit case



Transient Excitation

- Impulsive excitation by Shaker:



Modal Testing

(Lecture 14)

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RESPONSE FUNCTION MEASUREMENT TECHNIQUES

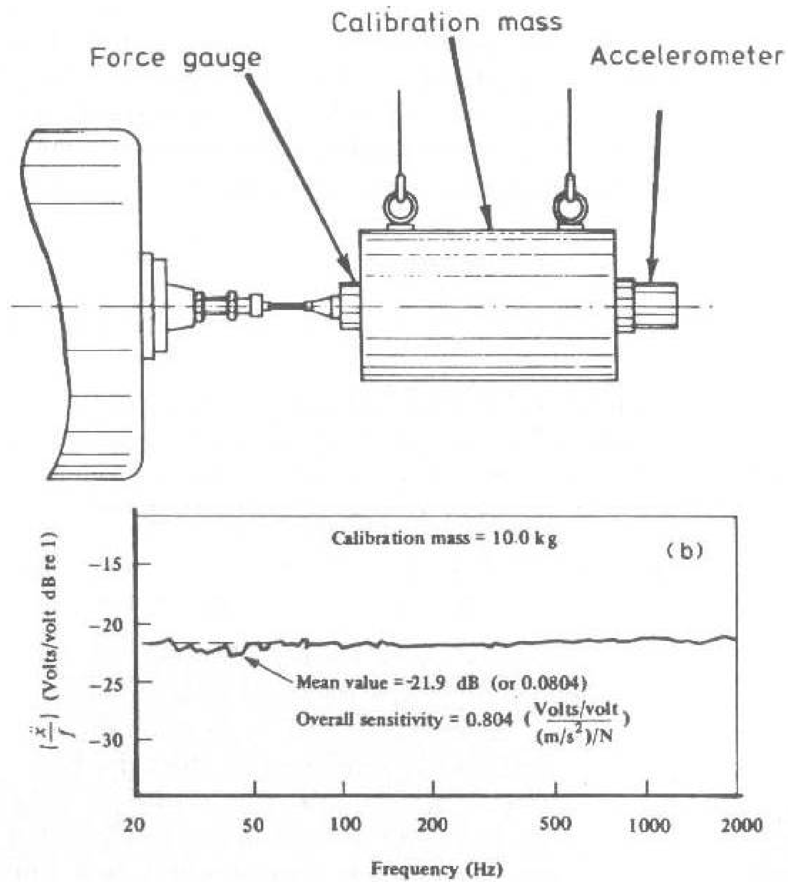
- 3.9 Calibration
- 3.10 Mass Cancellation
- 3.11 Rotational FRF Measurement
- 3.12 Measurement on Nonlinear Structures
  - Effects of Different Excitations
  - Level Control in FRF Measurement

Calibration

- In all measurement systems it is necessary to calibrate the equipment.
- There should be two levels of calibration:
  - Absolute calibration of individual transducers
  - The overall sensitivity of instrumentation system

Calibration

- The overall system calibration
  - The scale factor should be checked against computed factor using manufacturers stated sensitivity
  - Should be carried out before & after each test



Mass Cancellation

- Near resonance the actual applied force becomes very small and is thus very prone to inaccuracy.
- Some applied mass is used to move additional transducer mass

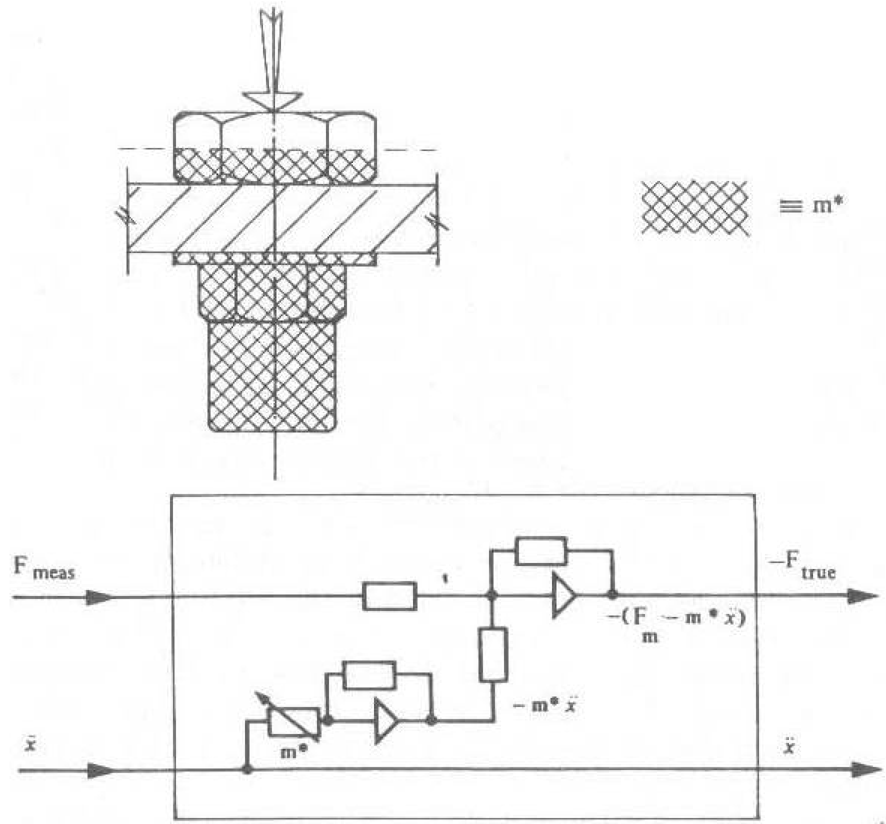
$$f_T = f_M - m^* \ddot{x}$$

$$\alpha_T = \frac{\ddot{X}}{F_T} \rightarrow required$$

$$\alpha_M = \frac{\ddot{X}}{F_M} \rightarrow measured$$

Mass Cancellation

- Added mass to be cancelled and the typical analogue circuit
- At deriving point a relation between measured and required FRF's can be obtained



Mass Cancellation

$$\mathrm{Re}(F_T) = \mathrm{Re}(F_M) - m^* \,\mathrm{Re}(\ddot{X})$$

$$\mathrm{Im}(F_T) = \mathrm{Im}(F_M) - m^* \,\mathrm{Im}(\ddot{X})$$

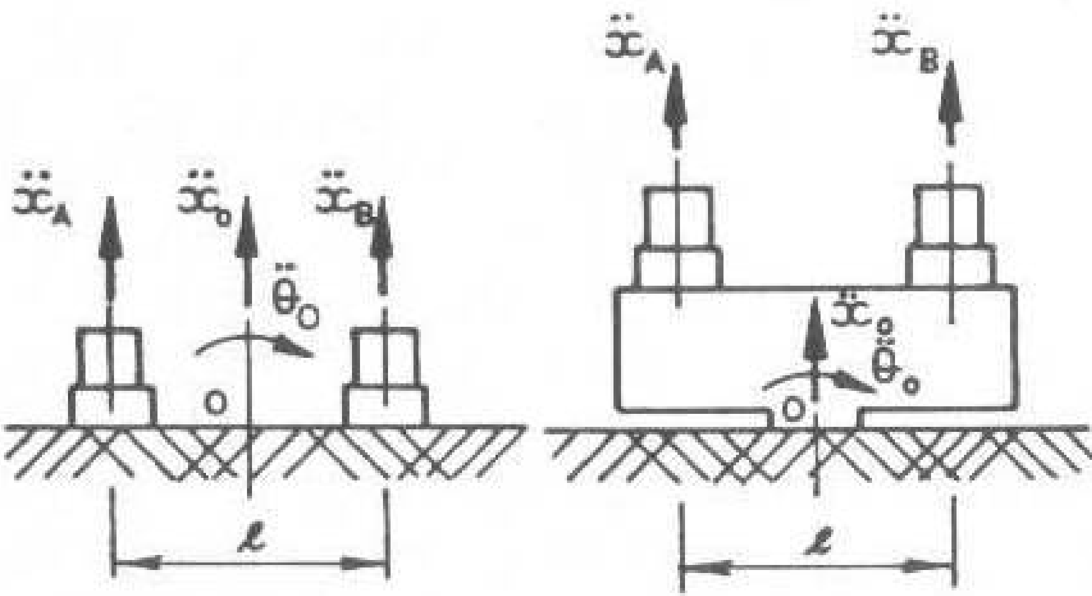
or

$$\mathrm{Re}(1/\alpha_T) = \mathrm{Re}(1/\alpha_M) - m^*$$

$$\mathrm{Im}(1/\alpha_T) = \mathrm{Im}(1/\alpha_M)$$

Rotational FRF Measurement

- Measurement of rotational FRFs using two or more transducers:



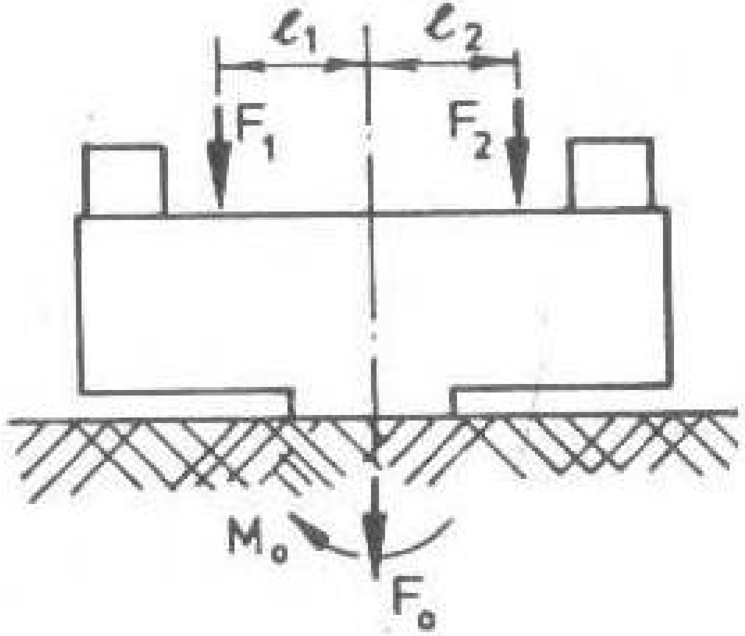
$$x_o = \frac{x_A + x_B}{2}$$

$$\theta_o = \frac{\theta_A + \theta_B}{L}$$

Rotational FRF Measurement

- Application of moment excitation

$$\frac{X}{F}, \frac{X}{M}, \frac{\theta}{F}, \frac{\theta}{M}$$

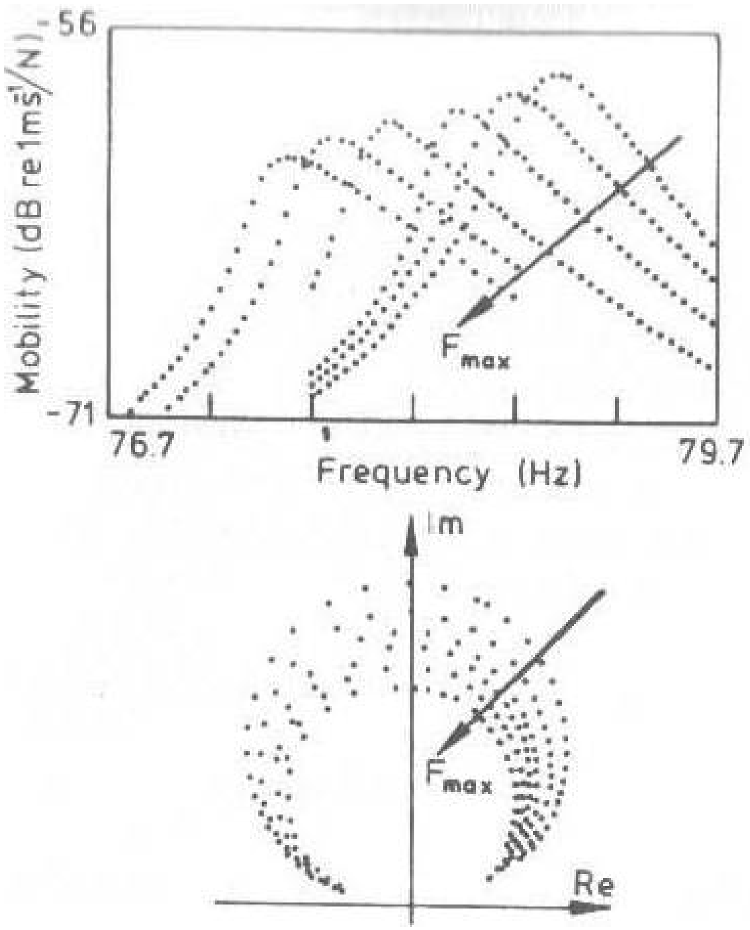


Measurement on Nonlinear Structures

- Many structures, especially in vicinity of resonances, behave in a nonlinear way:
  - Natural frequency varies with **position** and **strength of excitation**
  - **Distorted frequency** responses (near resonances)
  - Unstable or unrepeatable data

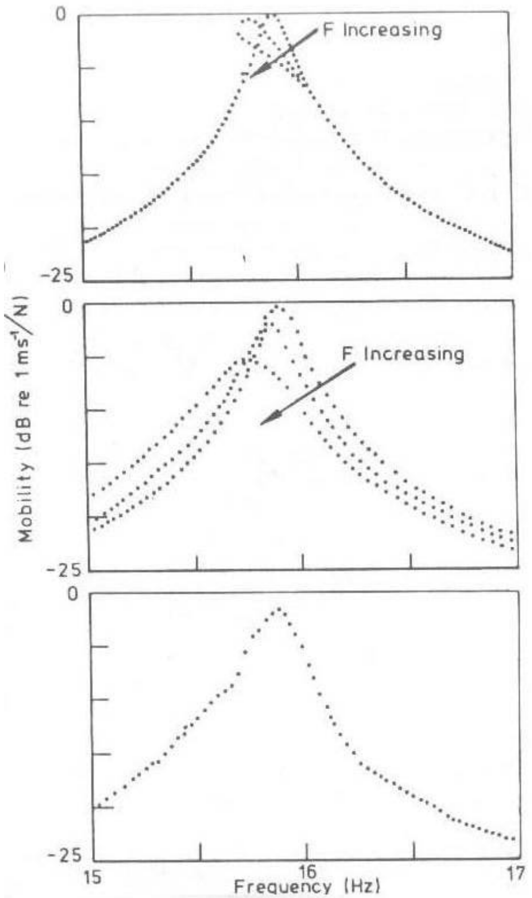
Measurement on Nonlinear Structures

- Examples of different nonlinear system response for different excitation levels
  - Softening effect
  - Increase in damping



Effects of Different Excitations

- FRF measurement on nonlinear system:
  - Sinusoidal Excitation
    - Compatible with theory
  - Random Excitation
    - Linearized system
  - Transient Excitation

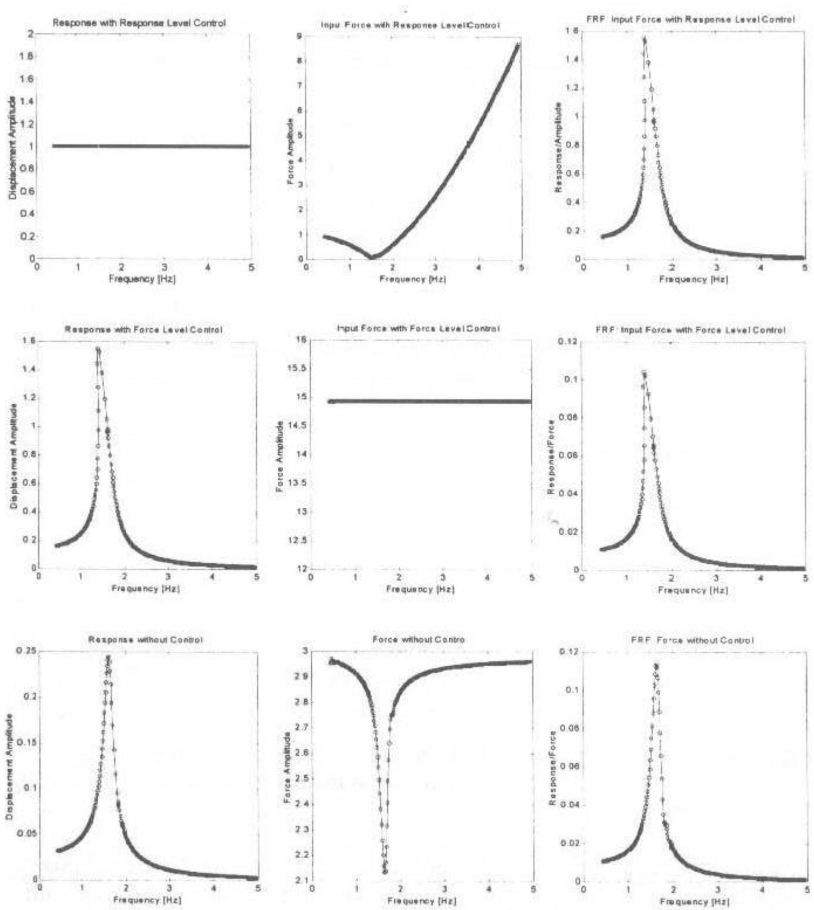


Effects of Different Excitations

- Most types of nonlinearity are amplitude dependent:
  - A linearized behaviour is observed when the response level is kept constant
  - The obtained linear model is valid for that particular vibration level

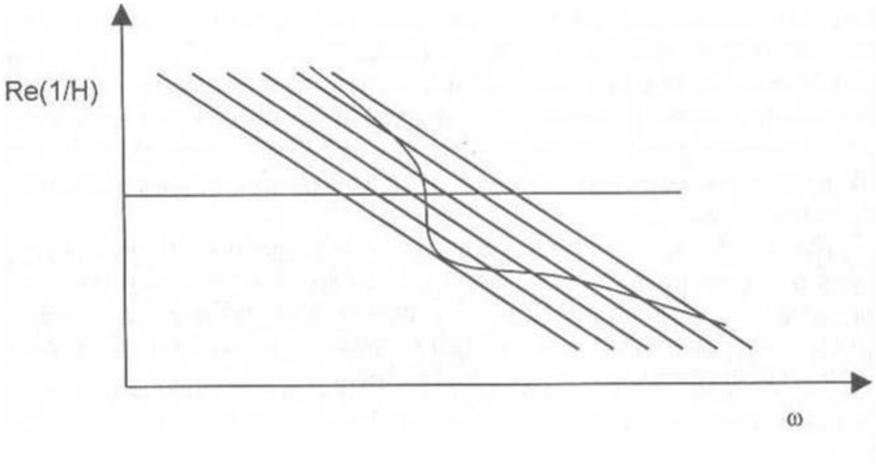
Level Control in FRF Measurement

- Response level control,
  - Best linear representation (nonlinearities are displacement dependent)
- Force level control
- Or no level control



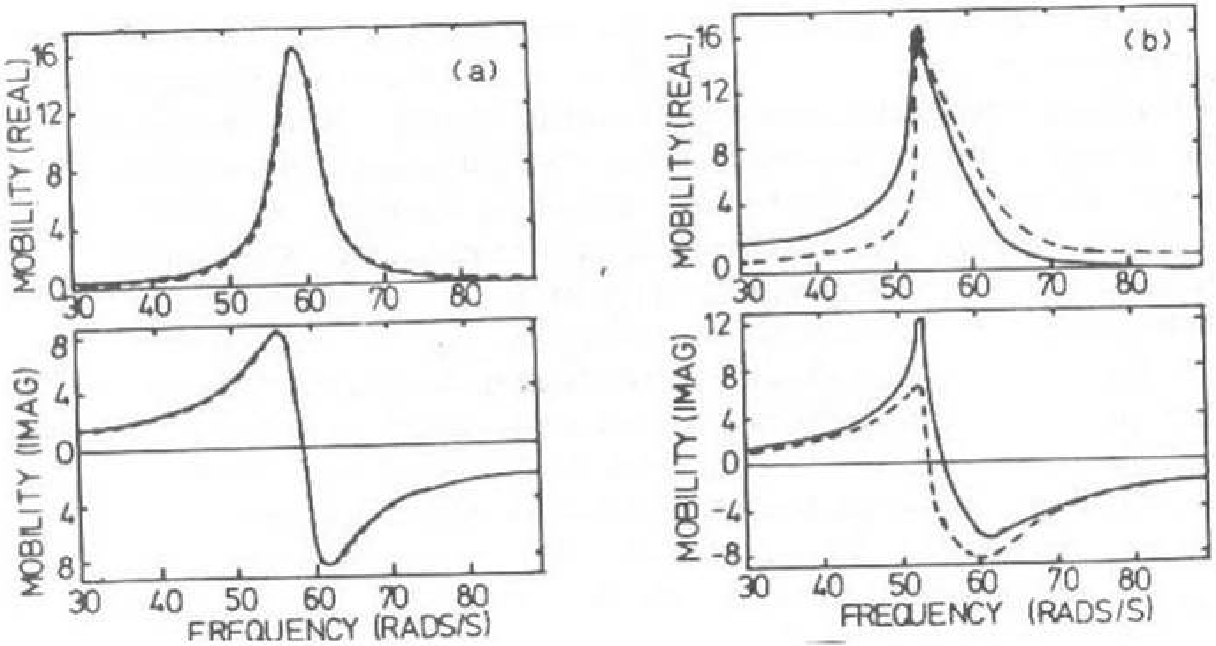
Level Control in FRF Measurement

- Inverse FRF plots for a SDOF
  - Real part is expected to be liner wrt frequency squared
  - Imaginary part should be linear/constant
  - Any deviation from the expected behaviour can be detected as nonlinearity in the system



Level Control in FRF Measurement

- Use of Hilbert transform to detect non-linearity
  - The Hilbert transform express the relations between real and imaginary parts of the Fourier Transform



Notes: Hilbert Transform

- The Hilbert transform express the relations between real and imaginary parts of the Fourier Transform
  - Fourier Transform is considered to map functions of time to functions of frequency and *vice versa*
  - Hilbert transform map functions of time or frequency to the same domain

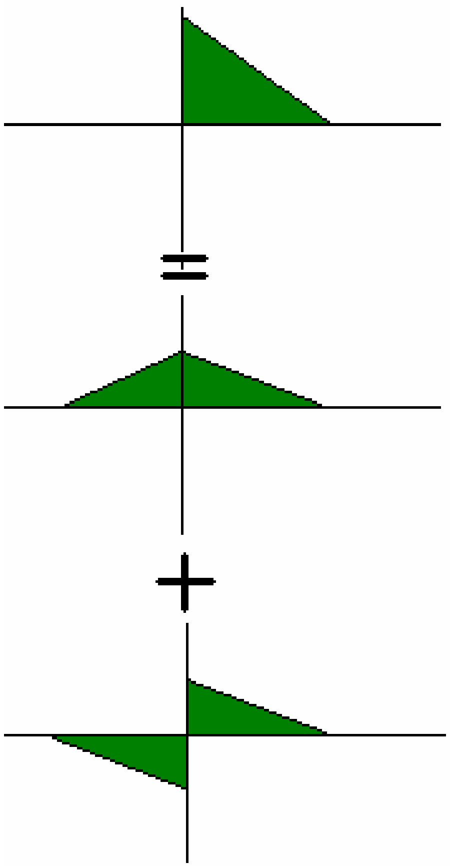
Notes: Hilbert Transform

- For causal functions:

$$g(t) = g_{even}(t) + g_{odd}(t),$$

$$g_{even}(t) = \begin{cases} g(|t|)/2, t > 0 \\ g(|t|)/2, t < 0 \end{cases}$$

$$g_{odd}(t) = \begin{cases} g(|t|)/2, t > 0 \\ -g(|t|)/2, t < 0 \end{cases}$$



Notes: Hilbert Transform

$$\operatorname{Re} G(\omega) = \Im\{g_{even}(t)\} = \Im\{g_{odd}(t) \times sign(t)\},$$

$$\operatorname{Im} G(\omega) = \Im\{g_{odd}(t)\} = \Im\{g_{even}(t) \times sign(t)\},$$

$$Since\ \Im\{sign(t)\} = \frac{-i}{\pi\omega}\ based\ on\ convolution\ theorm:$$

$$\operatorname{Re} G(\omega) = i \operatorname{Im} G(\omega) * \frac{-i}{\pi\omega},$$

$$\operatorname{Im} G(\omega) = \operatorname{Re} G(\omega) * \frac{-i}{\pi\omega}.$$

Modal Testing

(Lecture 15)

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Modal Parameter Extraction

- Introduction
- Preliminary checks of FRF data
  - Visual checks
  - Assessment of multiple-FRF data set using SVD
  - Mode indicator functions
- SDOF modal analysis methods
  - Peak amplitude method
  - Circle fit method
  - Inverse or line fit method

Introduction

- Some of the many available procedures for fitting a model to the measured data are discussed:
  - Their various advantages and limitations are explained,
  - No single method is best for all cases.
- This phase of the modal test procedure is often called ***modal parameter extraction*** or ***modal analysis***

Introduction

- Types of modal analysis:
  - Frequency domain (of FRFs)
  - Time domain (of Impulse Response Function)
- The analysis will be performed using
  - SDOF methods, and
  - MODF methods.

Introduction

- Another classification of methods relates to the number of FRFs used in the analysis:
  - Single-FRF methods, and
  - Multi-FRF methods:
    - Global methods which deals with SIMO data sets
    - and Polyreference which deals with MIMO data

Introduction

- Difficulty due to damping:
  - In practice we are obliged to make certain assumption about the damping model,
  - Significant errors can be incurred in the modal parameter estimates as a result of conflict between assumed and actual damping effects.
  - Decision on the issue of real and complex modes.

Preliminary checks of FRF data

- Low-frequency asymptotes,
  - Stiffness-like characteristics for grounded structures
  - Mass-line asymptotes for free structures
- High-frequency asymptotes,
  - Mass line or stiffness line
- Incidence of antiresonances
  - For a point FRF there must be a resonance after each antiresonance

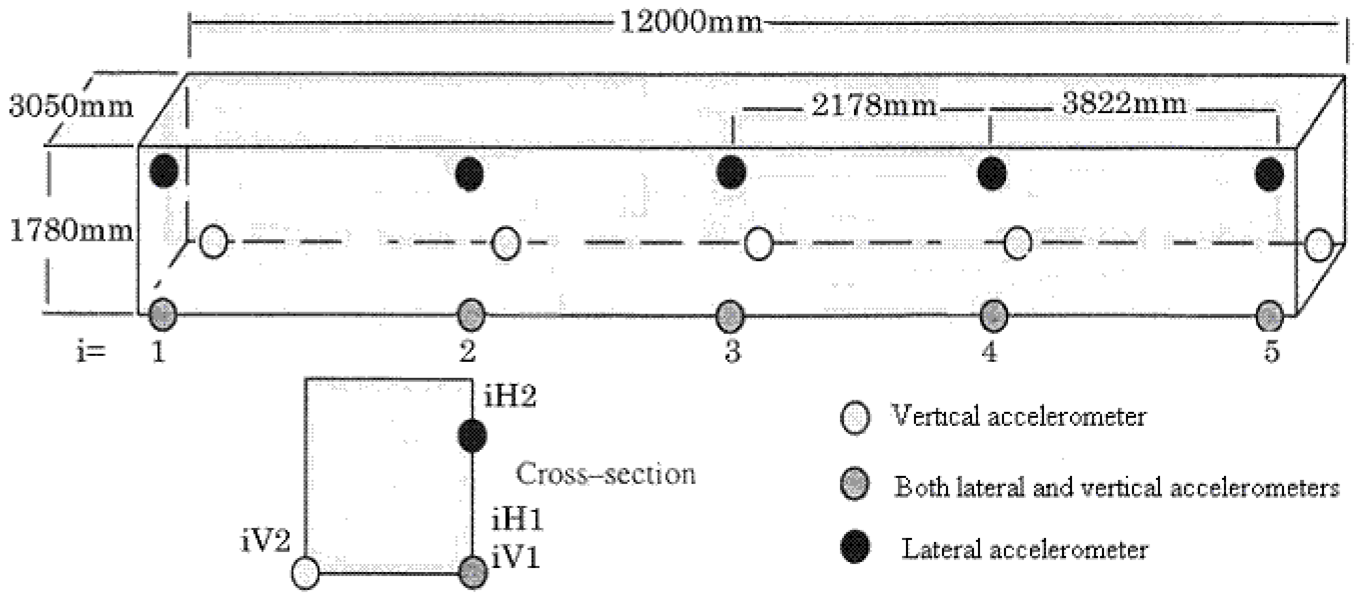
Preliminary checks of FRF data

- Mode Indicator Functions:
  - The Peak-Picking Method
    - Sum of amplitudes of all measured FRFs to locate the resonance points
  - The frequency-domain decomposition method
    - Defined by the SVD of the FRF matrix

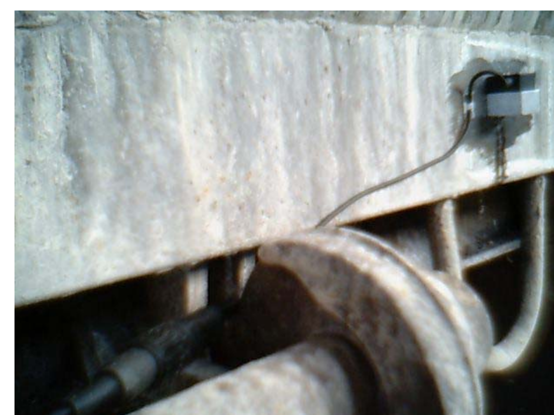
Case Study: MODES OF A RAILWAY VEHICLE

Case Study: Test set-up

Case Study: Sensor Locations



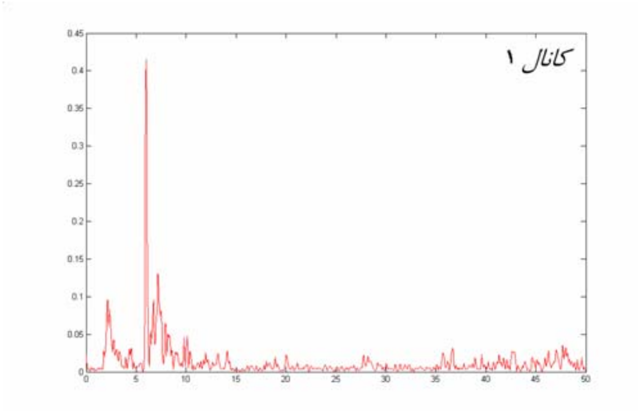
Case Study: Sensor Locations

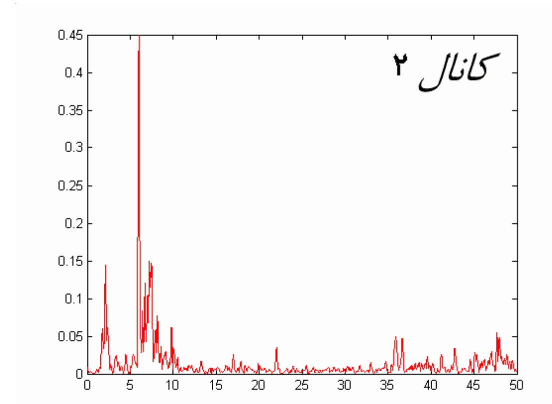


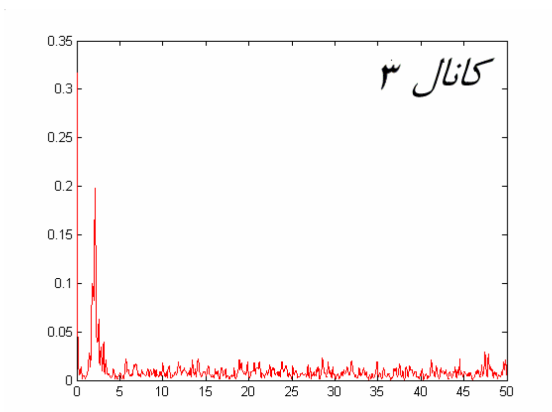
Case Study: Excitation

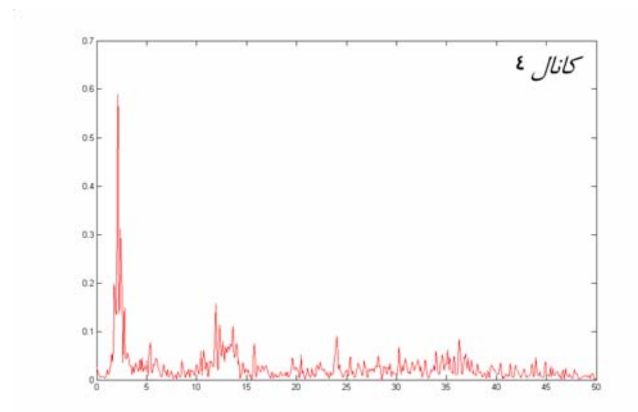
Case Study: Excitation

Case Study: Measurements



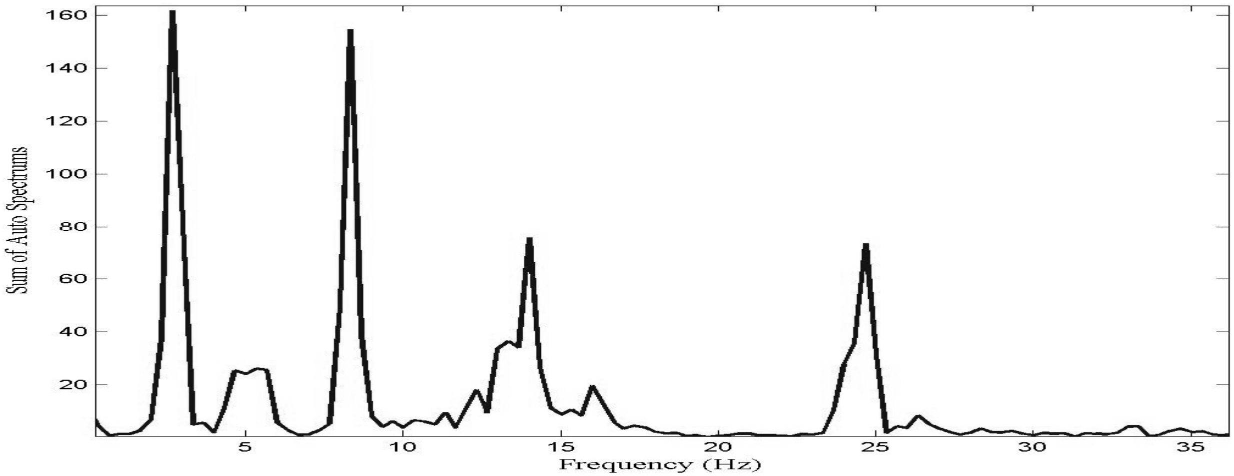






The Peak-Picking Method

- Sum of amplitudes of all measured FRFs to locate the resonance points



# Mode	1	2	3	4	5	6	7	8	9
(Hz) Frequency	2.67	4.67	5.33	8.33	12.33	13.33	14.00	16.00	24.67

The frequency-domain decomposition method

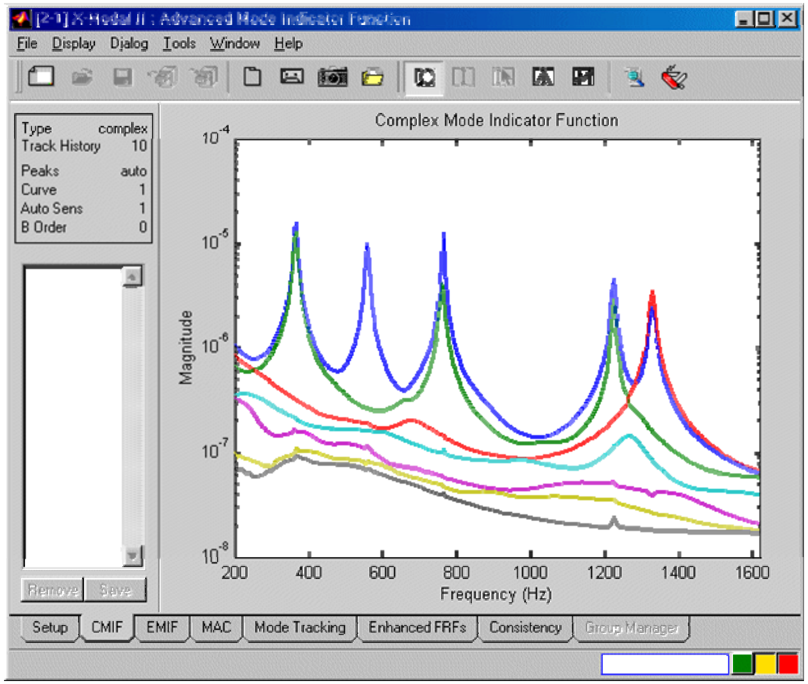
- A more advanced method consists of computing the Singular Value Decomposition of the spectrum matrix.
- The method is based on the fact that the transfer function or spectrum matrix evaluated at a certain frequency is only determined by neighboring modes.

The frequency-domain decomposition method

$$[H(\omega)] = \left[\{H_{11}(\omega)\} \quad \{H_{21}(\omega)\} \quad \dots \quad \{H_{np}(\omega)\}\right]$$

$$[H(\omega)] = [U(\omega)][\Sigma(\omega)][V(\omega)]^T$$

$$[MIF(\omega)] = [\Sigma(\omega)]^T [\Sigma(\omega)]$$



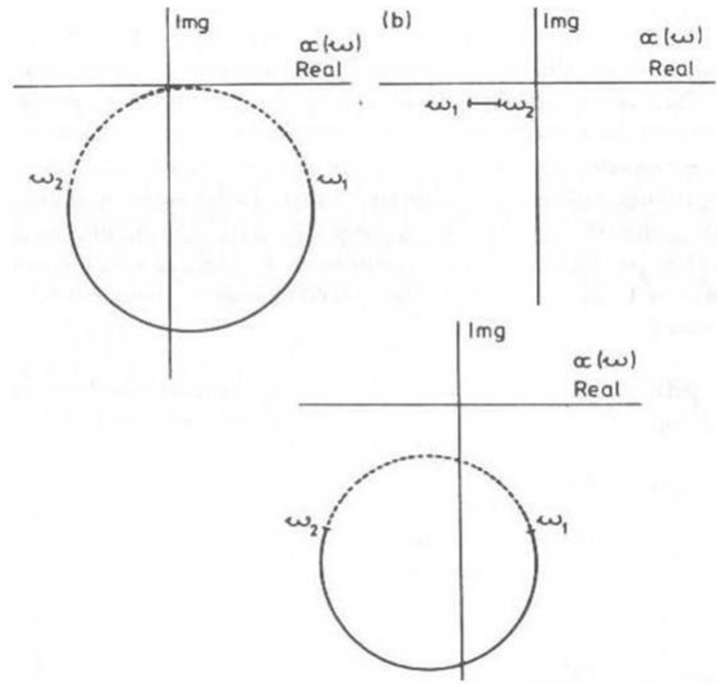
SDOF modal analysis methods

- The SDOF assumption

$$\alpha_{jk}(\omega)=\sum_{s=1}^{N}\frac{{}_{s}A_{jk}}{\omega_s^2-\omega+i\eta_s\omega_s^2}$$

$$\alpha_{jk}(\omega)=\frac{{}_{r}A_{jk}}{\omega_r^2-\omega+i\eta_r\omega_r^2}+\sum_{\substack{s=1\\ s\neq r}}^{N}\frac{{}_{s}A_{jk}}{\omega_s^2-\omega+i\eta_s\omega_s^2}$$

$$\alpha_{jk}(\omega)=\frac{{}_{r}A_{jk}}{\omega_r^2-\omega+i\eta_r\omega_r^2}+{}_{r}B_{jk}$$



SDOF modal analysis methods

- SDOF modal analysis methods
  - Peak amplitude method
  - Circle fit method
  - Inverse or line fit method

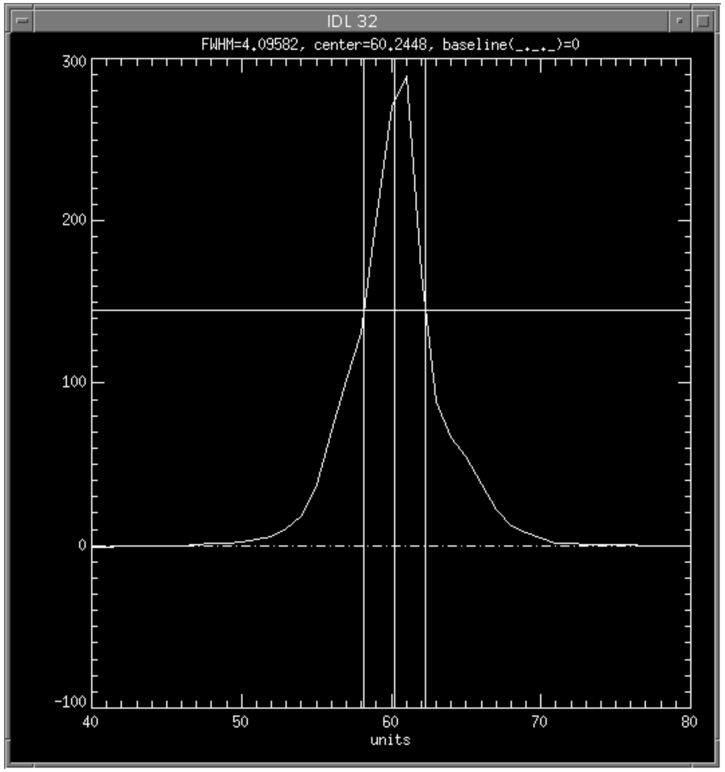
SDOF modal analysis methods: Peak Amplitude

- Individual resonance peaks are detected from the FRF
  - The frequency of the maximum responses is takes as the natural frequency of that mode,
  - The peak amplitude and the half power points are determined,

SDOF modal analysis methods: Peak Amplitude

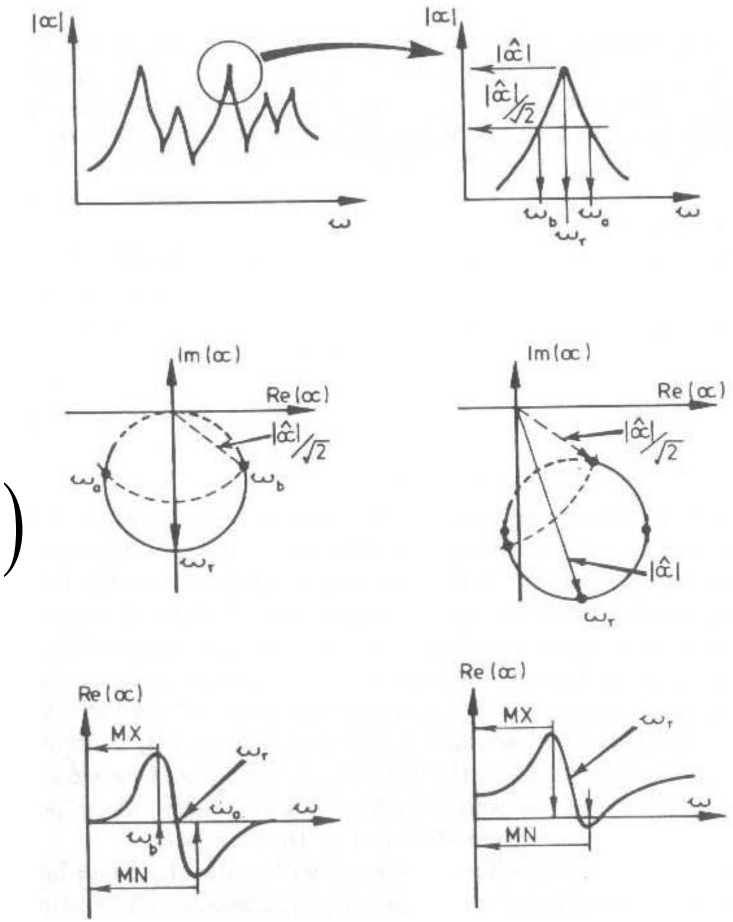
$$knowns \Rightarrow \begin{cases} \omega_r \\ \left|\hat{H}\right| \\ \omega_a, \omega_b \end{cases} ,$$

$$then \Rightarrow \begin{cases} \eta_r = \dfrac{\omega_a^2 - \omega_b^2}{2\omega_r^2} = \dfrac{\omega_a - \omega_b}{\omega_r}, \qquad 2\zeta_r = \eta_r \\ \left|\hat{H}\right| = \dfrac{A_r}{\eta_r \omega_r^2}, \qquad A_r = \eta_r \omega_r^2 \left|\hat{H}\right| \end{cases}$$



SDOF modal analysis methods: Peak Amplitude

- Another estimate for modal residue:



$$\left|\hat{H}\right| = \left(\left|\max(\mathrm{Re})\right| + \left|\min(\mathrm{Re})\right|\right)$$

$$A_r = \eta_r \omega_r^2 \left(\left|\max(\mathrm{Re})\right| + \left|\min(\mathrm{Re})\right|\right)$$

Modal Testing

(Lecture 16)

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Modal Parameter Extraction

- Circle-fit method
  - Properties of the modal circle
  - Circle-fit analysis procedure
  - Interpretation of damping plots

Properties of the modal circle

- Assuming a system with structural damping the basic function to deal with is:

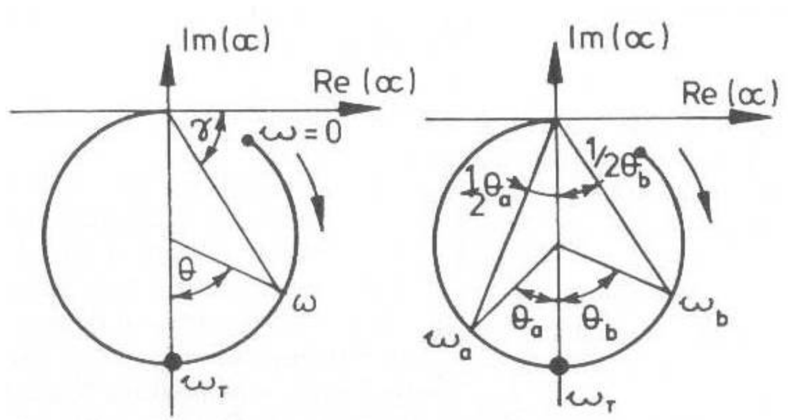
$$\alpha(\omega) = \frac{{}_rA_{jk}}{\omega_r^2(1-(\omega^2/\omega_r^2)+i\eta_r)}$$

- Since the effect of modal constant is to scale the size and rotate the circle, we consider:

$$\alpha(\omega) = \frac{1}{\omega_r^2(1-(\omega^2/\omega_r^2)+i\eta_r)}$$

Properties of the modal circle

- Finding the natural frequency:



$$\tan\gamma = \frac{\eta_r}{1-(\omega/\omega_r)^2}, \quad \tan(90-\gamma) = \tan(\frac{\theta}{2}) = \frac{1-(\omega/\omega_r)^2}{\eta_r}$$

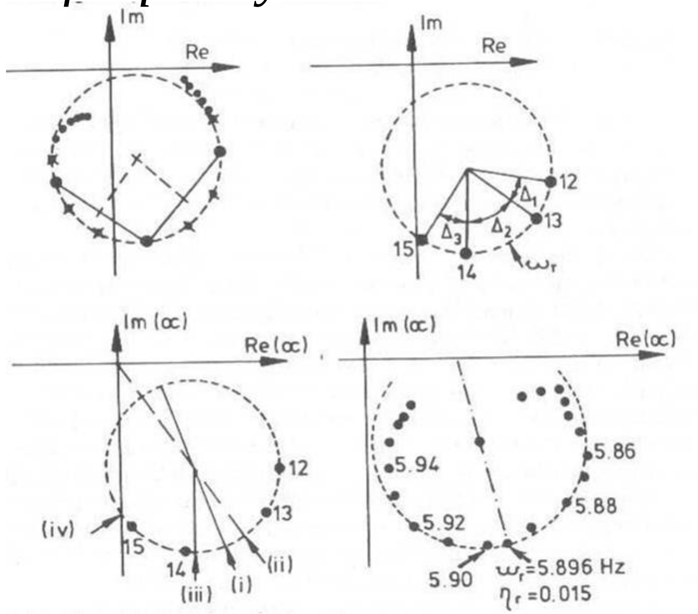
$$\Rightarrow \omega^2 = \omega_r^2(1-\eta_r \tan(\frac{\theta}{2})) \Rightarrow \frac{d\omega^2}{d\theta} = -\frac{\eta_r\omega_r^2}{2}\left(1+\left(\frac{1-(\omega/\omega_r)^2}{\eta_r}\right)^2\right)$$

Properties of the modal circle

$$\frac{d\omega^2}{d\theta}=-\frac{\eta_r\omega_r^2}{2}\left(1+\left(\frac{1-(\omega/\omega_r)^2}{\eta_r}\right)^2\right)\Rightarrow\frac{1}{(sweep\quad rate)}$$

$$\frac{d}{d\omega}\left(\frac{d\omega^2}{d\theta}\right)=0.@\quad\omega=\omega_r\Rightarrow Natural\ frequency$$

$$\left(\frac{d\theta}{d\omega^2}\right)_{\omega=\omega_r}=-\frac{2}{\eta_r\omega_r^2}\Rightarrow Damping$$



Properties of the modal circle

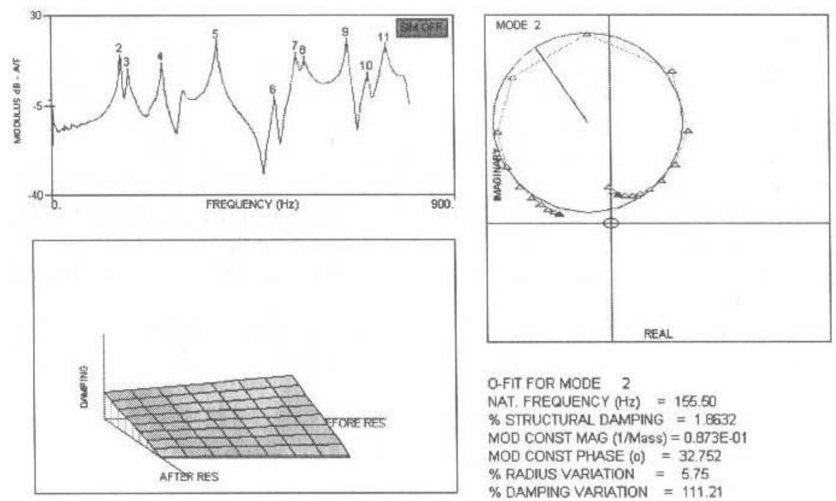
$$\begin{cases} \tan(\frac{\theta_b}{2}) = \dfrac{1-(\omega_b/\omega_r)^2}{\eta_r} \\ \tan(\frac{\theta_a}{2}) = \dfrac{(\omega_a/\omega_r)^2-1}{\eta_r} \end{cases} \Rightarrow \eta_r = \frac{\omega_a^2-\omega_b^2}{\omega_r^2(\tan(\frac{\theta_a}{2})+\tan(\frac{\theta_b}{2}))}$$

$$for\, \eta_r \le 2\% \ldots 3\%$$

$$\Rightarrow \eta_r = \frac{2(\omega_a-\omega_b)}{\omega_r(\tan(\frac{\theta_a}{2})+\tan(\frac{\theta_b}{2}))}$$

$$when\, \theta_a = \theta_b = 90^\circ$$

$$\Rightarrow \eta_r = \frac{\omega_a-\omega_b}{\omega_r}$$



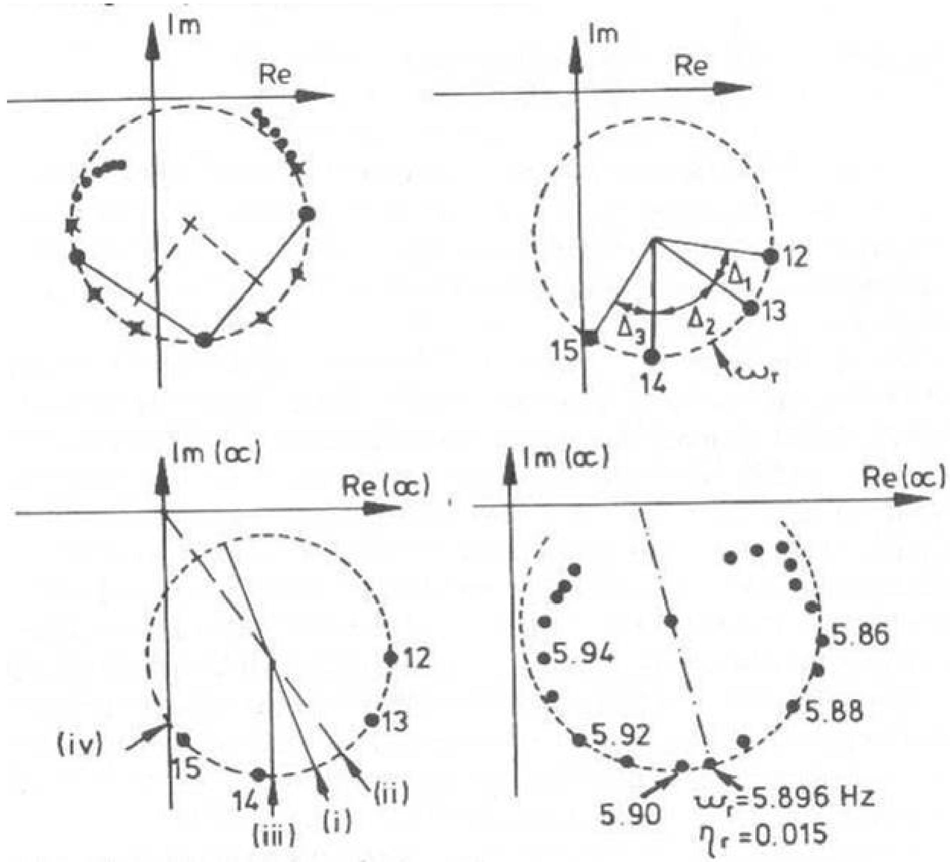
Properties of the modal circle

- The final property relates to the diameter of the circle (D):

$$ {}_rD_{jk} = \frac{\left| {}_rA_{jk} \right|}{\eta_r \omega_r^2} $$

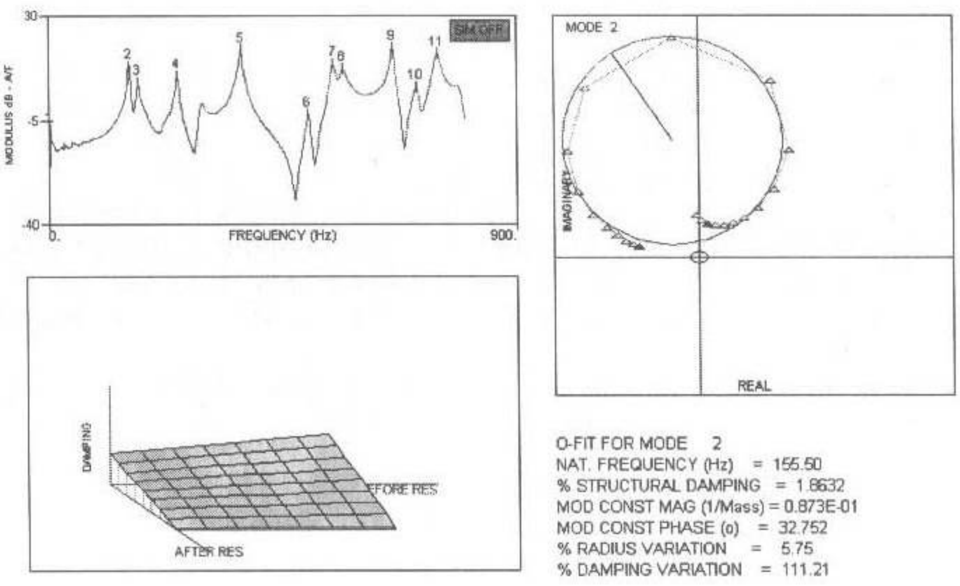
Circle-fit analysis procedure

- Select points to be used
- Fit circle, calculate quality of fit
- Locate natural frequency,
- …



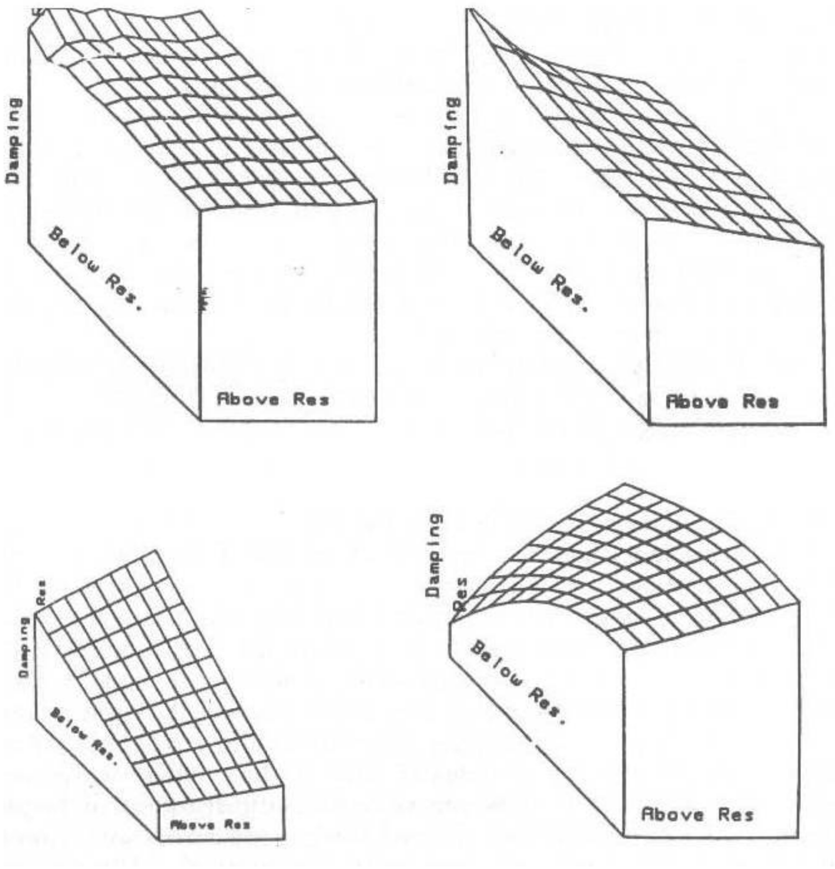
Circle-fit analysis procedure

- Obtain damping estimates
  - Calculate multiple damping estimate and scatter
- Determine modal constant module and argument.



Interpretation of damping plots

- Noise may contribute to the roughness of the surface.
- Systematic distortions due to:
  - Leakage
  - Erroneous estimates for natural frequency
  - Nonlinearity



Circle-fit

- Minimizing the algebraic distance:

$$(x+a)^2 + (y+b)^2 = R^2$$

$$x^2 + y^2 + Ax + By + C = 0.$$

$$\textit{Least Squares} \quad \textit{Solution}:$$

$$\begin{bmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n^2 + y_n^2 & x_n & y_n & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ A \\ B \\ C \end{Bmatrix} = 0.$$

Circle-fit

- Minimizing the geometric distance:

$$(x_1 + a)^2 + (x_2 + b)^2 = R^2$$

$$d_i^2 = \left(\left\| X_i - \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \right\| - u_3 \right)^2, Let \quad U = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\min \sum_{i=1}^{n} d_i(U)^2$$

Circle-fit

$$d = d_0 + \frac{\partial d_0}{\partial p}\Delta p + \cdots$$

$$\min\|d\| \approx \min\left\| d_0 + \frac{\partial d_0}{\partial p}\Delta p \right\|$$

$$\frac{\partial d_0}{\partial p} = \begin{bmatrix} \dfrac{u_1 - x_{11}}{\sqrt{(u_1 - x_{11})^2 + (u_2 - x_{12})^2}} & \dfrac{u_2 - x_{12}}{\sqrt{(u_1 - x_{11})^2 + (u_2 - x_{12})^2}} & -1 \\ \vdots & \vdots & \vdots \\ \dfrac{u_1 - x_{m1}}{\sqrt{(u_1 - x_{m1})^2 + (u_2 - x_{m2})^2}} & \dfrac{u_2 - x_{m2}}{\sqrt{(u_1 - x_{m1})^2 + (u_2 - x_{m2})^2}} & -1 \end{bmatrix}$$

Circle-fit

- Least-Squares Fitting of Circles and Ellipses By: Walter Gander, Gene H. Golub, and Rolf Strebel
- You may find it in ftpmech.iust.ac.ir

Home work 1

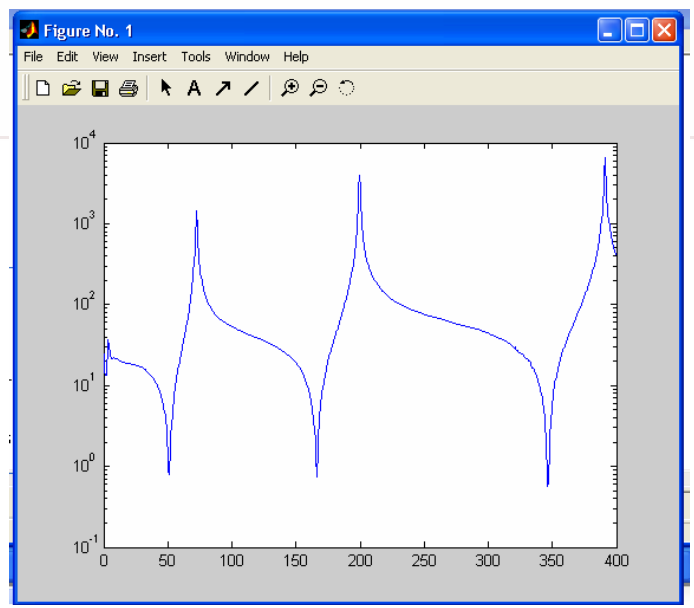
- Determine the modal properties of the beam tested in the lab
  - Frequency range of 0-400Hz
    - Natural frequencies
    - Damping (carpet plots)
    - Mode Shapes
- Due time 87/2/22

Importing the ASCII files

```
%     -1
%     58
% 24-Sep-106 14:14:03
% NONE
% 24-Sep-10 14:14:03
% EXCITATIONRESPONSE
% NONE
%     4          0    0          0 NONE                1   3 NONE                  1   3
%           5        801         1  0.00000E+00  5.00000E-01  0.00000E+00
%          18    0    0     0 NONE                NONE
%          12    0    0     0 NONE                NONE
%          13    0    0     0 NONE                NONE
%           0    0    0     0 NONE                NONE
clc;
FRF=[...
 -4.34000E-01  8.62000E-05 -1.73000E-01  2.08000E-01 -6.38000E-02  2.29000E-01
 -1.66000E-02  2.32000E-01 -6.17000E-03  2.34000E-01 -2.25000E-02  2.88000E-01
 -1.43000E-02  4.55000E-01  3.02000E-01  5.58000E-01  4.29000E-01  3.53000E-01
  4.30000E-01  2.62000E-01  4.29000E-01  1.84000E-01  3.97000E-01  1.33000E-01
  3.61000E-01  1.10000E-01  3.32000E-01  1.31000E-01  3.51000E-01  1.42000E-01
  3.59000E-01  1.29000E-01  3.65000E-01  1.18000E-01  3.70000E-01  1.05000E-01
```

Importing the ASCII files

```
  1.64000E+01  2.10000E+00  1.45000E+01  1.63000E+00  1.30000E+01  1.34000E+00
  1.18000E+01  1.13000E+00  1.08000E+01  9.63000E-01  1.00000E+01  8.35000E-01
  9.33000E+00  7.51000E-01  8.74000E+00  6.71000E-01  8.23000E+00  6.21000E-01
  7.79000E+00  5.67000E-01  7.40000E+00  5.41000E-01  7.06000E+00  4.96000E-01];
%      -1
FRF=[FRF(:,1)+i*FRF(:,2) FRF(:,3)+i*FRF(:,4) FRF(:,5)+i*FRF(:,6)];
FRF=FRF';
FRF=FRF(:);
semilogy([0:.5:400],abs(FRF)*180/pi);
```



Modal Testing

(Lecture 18)

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MDOF Modal Analysis in the Frequency Domain (SISO)

- In some cases the SDOF approach to modal analysis is simply inadequate or inappropriate:
  - closely-coupled modes,
    - the natural frequencies are very closely spaced, or
    - which have relatively heavy damping,
  - those with extremely light damping

MDOF Modal Analysis in the Frequency Domain (SISO)

- One step MDOF curve fitting methods:
  - Non-linear Least Squares Method
  - Rational Fraction Polynomial Method
  - A method particularly suited to very lightly damped structures
- Global Modal Analysis in Frequency Domain
  - Global Rational Fraction Polynomial Method
  - Global SVD Method

Non-linear Least Squares Method

$$H_{jk}(\omega_l) = H_l = \sum_{r=m_1}^{m_2} \frac{{}_rA_{jk}}{\omega_r^2 - \omega_l^2 + i\eta_r\omega_r^2} + \frac{1}{K_r} + \frac{1}{\omega_l^2 M_r}$$

$$\varepsilon_l = H_l^m - H_l$$

The difference between measurement and analytical model

$$E = \sum_{l=1}^{p} w_l \varepsilon_l^2, \qquad \frac{dE}{dq} = 0., \qquad q = {}_1A_{jk}, {}_2A_{jk}, {}_3A_{jk}, \cdots, \omega_1, etc$$

Non-linear Least Squares Method

- The set of obtained equations are nonlinear
  - No direct solution (iterative procedures)
  - Non-uniqueness of solution
  - Huge computational load

Rational Fraction Polynomial Method

$$H(\omega)=\sum_{r=1}^{N}\frac{{}_rA_{jk}}{\omega_r^2-\omega^2+2i\omega\omega_r\zeta_r}$$

$$H(\omega)=\frac{b_0+b_1(i\omega)+b_2(i\omega)^2+\cdots+b_{2N-1}(i\omega)^{2N-1}}{a_0+a_1(i\omega)+a_2(i\omega)^2+\cdots+a_{2N}(i\omega)^{2N}}$$

Rational Fraction Polynomial Method

Order of model is selected

$$e_k = \frac{b_0 + b_1(i\omega_k) + b_2(i\omega_k)^2 + \cdots + b_{2m-1}(i\omega_k)^{2m-1}}{a_0 + a_1(i\omega_k) + a_2(i\omega_k)^2 + \cdots + a_{2m}(i\omega_k)^{2m}} - H_k$$

or

$$e'_k = \left(b_0 + b_1(i\omega_k) + b_2(i\omega_k)^2 + \cdots + b_{2m-1}(i\omega_k)^{2m-1}\right) - H_k\left(a_0 + a_1(i\omega_k) + a_2(i\omega_k)^2 + \cdots + a_{2m}(i\omega_k)^{2m}\right)$$

Rational Fraction Polynomial Method

$$e'_k = \begin{Bmatrix} 1 & (i\omega_k) & (i\omega_k)^2 & \cdots & (i\omega_k)^{2m-1} \end{Bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{2m-1} \end{Bmatrix}$$

$$-H_k \begin{Bmatrix} 1 & (i\omega_k) & (i\omega_k)^2 & \cdots & (i\omega_k)^{2m-1} \end{Bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{2m-1} \end{Bmatrix} - a_{2m} H_k (i\omega_k)^{2m}$$

Rational Fraction Polynomial Method

- A set of linear equations using each individual measured FRF is formed.
- The unknowns a_i and b_i are obtained using a least square solution.
- The modal properties are extracted from obtained coefficients a_i and b_i.
- The analysis may repeat for a different model order.

Lightly Damped Structures

- In these structures it is easy to locate the natural frequencies,
  - Its accuracy is equal to the frequency resolution of the analyzer
- The damping ratio is assumed to be zero.
- The modal constants are obtained using curve fittings.

Lightly Damped Structures

$$H(\omega) = \sum_{r=1}^{N} \frac{{}_rA_{jk}}{\omega_r^2 - \omega^2}$$

The natural frequencies are known

$$\begin{Bmatrix} H(\Omega_1) \\ H(\Omega_2) \\ \vdots \\ \vdots \end{Bmatrix} = \begin{bmatrix} \left(\omega_1^2 - \Omega_1^2\right)^{-1} & \left(\omega_2^2 - \Omega_1^2\right)^{-1} & \cdots \\ \left(\omega_1^2 - \Omega_2^2\right)^{-1} & \left(\omega_2^2 - \Omega_2^2\right)^{-1} & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} {}_1A_{jk} \\ {}_2A_{jk} \\ \vdots \\ \vdots \end{Bmatrix}$$

Global Modal Analysis in Frequency Domain

- So far each measured FRF is curve fitted individually,
  - Multi-estimates for global parameters (natural frequencies and damping)
- Another way is to use measured FRF curves collectively.
  - Frequency and damping characteristics appear explicitly.

Global Rational Fraction Polynomial Method

- If we take several FRF's from the same structure then the denominator polynomial will be the same in every case.
- A natural extension of RFP method is to fit all n FRFs simultaneously
  - 2m-1 values of a_i and,
  - and n(2m-1) values of b_i

Global SVD Method

$$\{H(\omega)\}_k = \begin{Bmatrix} H_{1k}(\omega) \\ H_{2k}(\omega) \\ \vdots \\ H_{nk}(\omega) \end{Bmatrix}_{n\times 1}$$

$$= [\Phi]_{n\times N} \left[(i\omega - s_r)\right]^{-1}_{N\times N} \{\phi_k\}_{N\times 1} + \{R_k(\omega)\}_{N\times 1}$$

Global SVD Method

$$\{H(\omega)\}_k = [\Phi]_{n\times N}\left[(i\omega - s_r)\right]^{-1}_{N\times N}\{\phi_k\}_{N\times 1} + \{R_k\}$$

$$\{\dot{H}(\omega)\}_k = [\Phi]_{n\times N}[s_r]\left[(i\omega - s_r)\right]^{-1}_{N\times N}\{\phi_k\}_{N\times 1} + \{R_k\}$$

$$\{g_k(\omega)\} = \left[(i\omega - s_r)\right]^{-1}_{N\times N}\{\phi_k\}_{N\times 1}$$

$$\{H(\omega)\}_k = [\Phi]_{n\times N}\{g_k(\omega)\} + \{R_k\}$$

$$\{\dot{H}(\omega)\}_k = [\Phi]_{n\times N}[s_r]\{g_k(\omega)\} + \{R_k\}$$

Global SVD Method

$$\{\Delta H(\omega_i)\}_k = \{H(\omega_i)\}_k - \{H(\omega_{i+c})\}_k$$

$$\{\Delta H(\omega_i)\}_k = [\Phi]_{n\times N}\{\Delta g_k(\omega_i)\}$$

$$\{\Delta \dot{H}(\omega_i)\}_k = [\Phi]_{n\times N}[s_r]\{\Delta g_k(\omega_i)\}$$

Global SVD Method

- Consider data from several different frequencies to obtain frequencies and damping:

$$\left[\Delta H(\omega)_k\right]_{n\times L} = [\Phi]_{n\times N}\left[\Delta g_k(\omega)\right]_{N\times L}$$

$$\left[\Delta \dot{H}(\omega)_k\right]_{n\times L} = [\Phi]_{n\times N}[s_r]\left[\Delta g_k(\omega)\right]_{N\times L}$$

$$\left(\left[\Delta \dot{H}_k\right]^T - s_r\left[\Delta H_k\right]^T\right)\{z_r\} = 0, \quad [z] = [\Phi]^{+T}$$

Global SVD Method

- The eigen-problem is solved using the SVD.
- The rank of the FRF matrices and eigenvalues are obtained.
- Then the modal constants can be recovered from:

$$\begin{Bmatrix} H_{jk}(\omega_1) \\ H_{jk}(\omega_2) \\ \cdots \\ H_{jk}(\omega_{1L}) \end{Bmatrix}_{L\times 1} = \begin{bmatrix} (i\omega_1 - s_1)^{-1} & (i\omega_1 - s_2)^{-1} & \cdots \\ (i\omega_2 - s_1)^{-1} & (i\omega_2 - s_2)^{-1} & \cdots \\ \cdots & \cdots & \cdots \\ (i\omega_L - s_1)^{-1} & \cdots & (i\omega_L - s_m)^{-1} \end{bmatrix} \begin{Bmatrix} {}_1A_{jk} \\ {}_2A_{jk} \\ \cdots \\ {}_mA_{jk} \end{Bmatrix}$$

Modal Testing

(Lecture 18-1)

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MDOF Modal Analysis in the Time Domain

- The basic concept: Any Impulse Response Function can be expressed by a series of Complex Exponentials

$$h_{jk}(t)=\sum_{r=1}^{2N} {}_r A_{jk} e^{s_r t}; \quad s_r=\omega_r\left(-\zeta_r+i\sqrt{1-\zeta_r^2}\right)$$

- The Complex Exponential Series contain the eigenvalues and eigenvectors information.
- The IRF is obtained by taking inverse Fourier transform of the measured FRF.

Complex Exponential Method

$$FRF \Rightarrow \alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{{}_rA_{jk}}{i\omega - s_r} + \frac{{}_rA_{jk}^*}{i\omega - s_r^*}$$

$$or: \quad \alpha_{jk}(\omega) = \sum_{r=1}^{2N} \frac{{}_rA_{jk}}{i\omega - s_r}$$

$$IRF \Rightarrow \qquad h_{jk}(t) = \sum_{r=1}^{2N} {}_rA_{jk} e^{s_r t}$$

Complex Exponential Method (Single FRF)

$$h(t)=\sum_{r=1}^{2N} A_r e^{s_r t} \Rightarrow h_l=\sum_{r=1}^{2N} A_r e^{s_r l\Delta t}=\sum_{r=1}^{2N} A_r V_r^l$$

$$\begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_q \end{Bmatrix}=\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^q & V_2^q & \cdots & \cdots & V_{2N}^q \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{Bmatrix}^T \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_q \end{Bmatrix} = \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{Bmatrix}^T \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^q & V_2^q & \cdots & \cdots & V_{2N}^q \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

$$\sum_{i=0}^{q} \beta_i h_i = \sum_{j=1}^{2N} A_j \left(\sum_{i=0}^{q} \beta_i V_j^i \right)$$

Complex Exponential Method (Single FRF)

- The β_i are selected to be coefficients of the polynomial:

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \cdots + \beta_q V^q = 0.$$

$$Set: q = 2N \Rightarrow \sum_{i=0}^{q} \beta_i h_i = \sum_{j=1}^{2N} A_j \left(\sum_{i=0}^{q} \beta_i V_j^i \right) \Rightarrow \begin{cases} \sum_{i=0}^{2N} \beta_i V_j^i = 0, \\ \sum_{i=0}^{2N} \beta_i h_i = 0. \end{cases}$$

$$\sum_{i=0}^{2N-1} \beta_i h_i = -h_{2N}$$

Complex Exponential Method (Single FRF)

$$\sum_{i=0}^{2N-1} \beta_i h_i = -h_{2N}$$

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{2N-1} \\ h_1 & h_2 & h_3 & \cdots & h_{2N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{2N-1} & h_{2N} & h_{2N+1} & \cdots & h_{4N-2} \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{2N-1} \end{Bmatrix} = -\begin{Bmatrix} h_{2N} \\ h_{2N+1} \\ \vdots \\ \vdots \\ h_{4N-1} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

- The values $V_i = e^{s_r \Delta t}$ and A_i are obtained from:

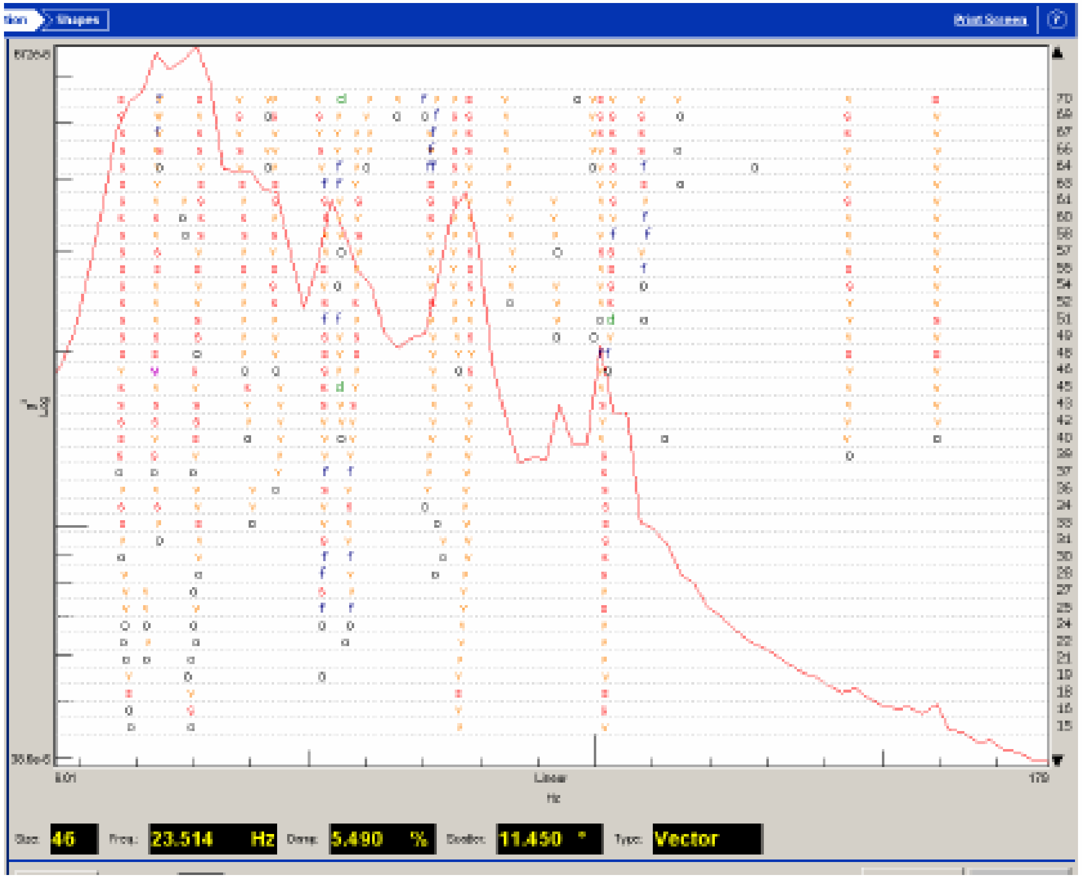
$$\beta_0 + \beta_1 V + \beta_2 V^2 + \cdots + \beta_{2N} V^{2N} = 0.$$

$$\begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{2N-1} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^{2N-1} & V_2^{2N-1} & \cdots & \cdots & V_{2N}^{2N-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

- Implementation Procedure:
  - Order of modal model is selected,
  - Modal model is identified using the defined steps in previous slides,
  - FRF is regenerated from modal information and compared with the measured FRF
  - The procedure repeated using another order for the modal model until stable results are obtained.

Stabilization Diagram



Global Analysis in Time Domain (Ibrahim Time Domain Method)

- The basic concept is to obtain a unique set of modal parameters from a set of vibration measurements:
  - Scaled (mass normalized) mode shapes when the force is known,
  - Un-scaled mode shapes when the force is not measured.

Ibrahim Time Domain Method

$$x_i(t)=\sum_{r=1}^{2m}\psi_{ir}e^{s_r t}$$

$$\begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_q) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_q) \\ \vdots & \vdots & \cdots & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_q) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1,2m} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,2m} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{n,2m} \end{bmatrix} \times \begin{bmatrix} e^{s_1 t_1} & \cdots & \cdots & e^{s_1 t_q} \\ e^{s_2 t_1} & \cdots & \cdots & e^{s_2 t_q} \\ \vdots & \cdots & \cdots & \vdots \\ e^{s_{2m} t_1} & \cdots & \cdots & e^{s_{2m} t_q} \end{bmatrix}$$

$$[X]=[\Psi]\times[\Lambda]$$

Ibrahim Time Domain Method

- A 2nd set of eqns:

$$x_i(t_l + \Delta t) = \sum_{r=1}^{2m} \psi_{ir} e^{s_r (t_l + \Delta t)}$$

$$= \sum_{r=1}^{2m} \left(\psi_{ir} e^{s_r \Delta t}\right) e^{s_r t_l} = \sum_{r=1}^{2m} \hat{\psi}_{ir} e^{s_r t_l}$$

$$\left\lfloor \hat{X} \right\rfloor = \left\lfloor \hat{\Psi} \right\rfloor \times [\Lambda]$$

Ibrahim Time Domain Method

$$[A]\times[\Psi]=\left[\hat{\Psi}\right]$$

$$\left.\begin{aligned}[X]&=[\Psi]\times[\Lambda]\\ \left[\hat{X}\right]&=\left[\hat{\Psi}\right]\times[\Lambda]\end{aligned}\right\}\Rightarrow[A]\times[X]=\left[\hat{X}\right]$$

$$[A]=\left[\hat{X}\right]\times[X]^{+}$$

Ibrahim Time Domain Method

$$[A]\{\psi_r\} = e^{s_r \Delta t}\{\psi_r\}$$

- Eigenvectors of matrix [A] are the mode shapes,
- The natural frequencies and damping ratios are obtained from eigenvalues of [A].

Modal Testing

(MDOF Modal Analysis in the Time Domain)

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MDOF Modal Analysis in the Time Domain

- The basic concept: Any Impulse Response Function can be expressed by a series of Complex Exponentials

$$h_{jk}(t) = \sum_{r=1}^{2N} {}_r A_{jk} e^{s_r t}; \quad s_r = \omega_r \left(-\zeta_r + i\sqrt{1-\zeta_r^2} \right)$$

- The Complex Exponential Series contain the eigenvalues and eigenvectors information.
- The IRF is obtained by taking inverse Fourier transform of the measured FRF.

Complex Exponential Method

(CE)

Complex Exponential Method

$$FRF \Rightarrow \alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{{}_rA_{jk}}{i\omega - s_r} + \frac{{}_rA_{jk}^*}{i\omega - s_r^*}$$

$$or: \quad \alpha_{jk}(\omega) = \sum_{r=1}^{2N} \frac{{}_rA_{jk}}{i\omega - s_r}$$

$$IRF \Rightarrow \qquad h_{jk}(t) = \sum_{r=1}^{2N} {}_rA_{jk} e^{s_r t}$$

Complex Exponential Method (Single FRF)

$$h(t)=\sum_{r=1}^{2N} A_r e^{s_r t} \Rightarrow h_l=\sum_{r=1}^{2N} A_r e^{s_r l\Delta t}=\sum_{r=1}^{2N} A_r V_r^{\,l}$$

$$\begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_q \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^q & V_2^q & \cdots & \cdots & V_{2N}^q \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{Bmatrix}^T \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_q \end{Bmatrix} = \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{Bmatrix}^T \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^q & V_2^q & \cdots & \cdots & V_{2N}^q \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

$$\sum_{i=0}^{q} \beta_i h_i = \sum_{j=1}^{2N} A_j \left(\sum_{i=0}^{q} \beta_i V_j^i \right)$$

Complex Exponential Method (Single FRF)

- The β_i are selected to be coefficients of the polynomial:

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \cdots + \beta_q V^q = 0.$$

$$Set: q = 2N \Rightarrow \sum_{i=0}^{q} \beta_i h_i = \sum_{j=1}^{2N} A_j \left(\sum_{i=0}^{q} \beta_i V_j^i \right) \Rightarrow \begin{cases} \sum_{i=0}^{2N} \beta_i V_j^i = 0, \\ \sum_{i=0}^{2N} \beta_i h_i = 0. \end{cases}$$

$$\sum_{i=0}^{2N-1} \beta_i h_i = -h_{2N}$$

Complex Exponential Method (Single FRF)

$$\sum_{i=0}^{2N-1} \beta_i h_i = -h_{2N}$$

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{2N-1} \\ h_1 & h_2 & h_3 & \cdots & h_{2N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{2N-1} & h_{2N} & h_{2N+1} & \cdots & h_{4N-2} \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{2N-1} \end{Bmatrix} = -\begin{Bmatrix} h_{2N} \\ h_{2N+1} \\ \vdots \\ \vdots \\ h_{4N-1} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

- The values $V_i = e^{s_r \Delta t}$ and A_i are obtained from:

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \cdots + \beta_{2N} V^{2N} = 0.$$

$$\begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{2N-1} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^{2N-1} & V_2^{2N-1} & \cdots & \cdots & V_{2N}^{2N-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix}$$

Complex Exponential Method (Single FRF)

- Implementation Procedure:
  - Order of modal model is selected,
  - Modal model is identified using the defined steps in previous slides,
  - FRF is regenerated from modal information and compared with the measured FRF
  - The procedure repeated using another order for the modal model until stable results are obtained.

The Least Squares Complex Exponential Method

(LSCE)

Dr H Ahmadian,
Modal Testing Lab,
IUST

The Least Squares Complex Exponential Method (LSCE)

- The LSCE is the extension of CE to a global procedure.
- It processes several IRF's obtained using SIMO method.
- The coefficients β that provide the solution of characteristic polynomial are global quantities.

The Least Squares Complex Exponential Method (LSCE)

One Typical IRF

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{2N-1} \\ h_1 & h_2 & h_3 & \cdots & h_{2N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{2N-1} & h_{2N} & h_{2N+1} & \cdots & h_{4N-2} \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{2N-1} \end{Bmatrix} = -\begin{Bmatrix} h_{2N} \\ h_{2N+1} \\ \vdots \\ \vdots \\ h_{4N-1} \end{Bmatrix}, \ or \ [h]_q \{\beta\} = \{h'\}_q$$

Extending to all measured IRFs

$$\begin{bmatrix} [h]_1 \\ [h]_2 \\ \vdots \\ [h]_p \end{bmatrix} \{\beta\} = \begin{Bmatrix} \{h'\}_1 \\ \{h'\}_2 \\ \vdots \\ \{h'\}_q \end{Bmatrix}, \ or \ [h_G]\{\beta\} = \{h'_G\} \Rightarrow \{\beta\} = \left([h_G]^T [h_G]\right)^{-1} [h_G]^T \{h'_G\}$$

The PolyReference Complex Exponential Method (PRCE)

The PolyReference Complex Exponential Method (PRCE)

- Constitutes the extension of LSCE to MIMO.
- A general and automatic way of analyzing dynamics of a structure.
- MIMO test method overcomes the problem of not exciting some modes as usually happens in SIMO.

The PolyReference Complex Exponential Method (PRCE)

Considering q input reference points:

$$h_{j1}(t) = \sum_{r=1}^{2N} {}_rA_{j1} e^{s_r t}$$

$$h_{j2}(t) = \sum_{r=1}^{2N} {}_rA_{j2} e^{s_r t}$$

$$\vdots \qquad \vdots$$

$$h_{jq}(t) = \sum_{r=1}^{2N} {}_rA_{jq} e^{s_r t}$$

$${}_rA_{jl} = Q_r \phi_{jr} \phi_{lr}$$

$${}_rA_{jk} = {}_rW_{kl} \; {}_rA_{jl}$$

$${}_rW_{kl} = \frac{\phi_{kr}}{\phi_{lr}}$$

Modal Participation factor

$$h_{j1}(t) = \sum_{r=1}^{2N} {}_rA_{j1} e^{s_r t}$$

$$h_{j2}(t) = \sum_{r=1}^{2N} {}_rW_{21} \; {}_rA_{j1} e^{s_r t}$$

$$\vdots \qquad \vdots$$

$$h_{jq}(t) = \sum_{r=1}^{2N} {}_rW_{q1} \; {}_rA_{j1} e^{s_r t}$$

The PolyReference Complex Exponential Method (PRCE)

$$h_{j1}(t) = \sum_{r=1}^{2N} {}_rA_{j1} e^{s_r t}$$

$$h_{j2}(t) = \sum_{r=1}^{2N} {}_rW_{21} \; {}_rA_{j1} e^{s_r t} \Rightarrow \{h_j(t)\} = [W][e^{\Lambda t}]\{A_{j1}\}$$

$$\vdots \qquad\qquad \vdots$$

$$h_{jq}(t) = \sum_{r=1}^{2N} {}_rW_{q1} \; {}_rA_{j1} e^{s_r t}$$

$$\begin{Bmatrix} h_{j1}(t) \\ h_{j2}(t) \\ \vdots \\ h_{jq}(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ {}_1W_{21} & {}_2W_{21} & & {}_{2N}W_{21} \\ \vdots & \vdots & \cdots & \vdots \\ {}_1W_{q1} & {}_2W_{q1} & & {}_{2N}W_{q1} \end{bmatrix} \begin{bmatrix} e^{s_1 t} & 0 & \cdots & 0 \\ 0 & e^{s_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{s_{2N} t} \end{bmatrix} \begin{Bmatrix} {}_1A_{j1} \\ {}_2A_{j1} \\ \vdots \\ {}_{2N}A_{j1} \end{Bmatrix}$$

The PolyReference Complex Exponential Method (PRCE)

$$\left\{h_j(0)\right\} = [W]\left\{A_{j1}\right\}$$
$$\left\{h_j(\Delta t)\right\} = [W][V]\left\{A_{j1}\right\} \qquad , \qquad [V] = \left[e^{\Lambda \Delta t}\right]$$
$$\vdots$$
$$\left\{h_j(L\Delta t)\right\} = [W][V]^L\left\{A_{j1}\right\}$$

$$[\beta_0] + [\beta_1][W][V] + [\beta_2][W][V]^2 + \cdots + [\beta_L][W][V]^L = [0], \; L\,q \geq 2N$$

The PolyReference Complex Exponential Method (PRCE)

$$[\beta_0]\{h_j(0)\}=[\beta_0][W]\{A_{j1}\}$$

$$[\beta_1]\{h_j(\Delta t)\}=[\beta_1][W][V]\{A_{j1}\}$$

$$[\beta_2]\{h_j(\Delta t)\}=[\beta_2][W][V]^2\{A_{j1}\}$$

$$\vdots$$

$$[\beta_L]\{h_j(L\Delta t)\}=[\beta_L][W][V]^L\{A_{j1}\}$$

$$\sum_{k=0}^{L}[\beta_k]\{h_j(k\Delta t)\}=\sum_{k=0}^{L}[\beta_k][W][V]^k\{A_{j1}\}$$

The PolyReference Complex Exponential Method (PRCE)

$$\sum_{k=0}^{L}[\beta_k]\{h_j(k\Delta t)\}=0,\quad [\beta_L]=[I]$$

$$\sum_{k=0}^{L-1}[\beta_k]\{h_j(k\Delta t)\}=-\{h_j(L\Delta t)\}$$

$$[[\beta_0]\quad[\beta_1]\quad\cdots\quad[\beta_{L-1}]]\begin{bmatrix}\{h_j(0)\} & \{h_j(\Delta t)\} & \{h_j((N_t-1)\Delta t)\}\\ \{h_j(\Delta t)\} & \{h_j(2\Delta t)\} & \{h_j(N_t\Delta t)\}\\ & & \\ \{h_j((L-1)\Delta t)\} & \{h_j(L\Delta t)\} & \{h_j((L+N_t-2)\Delta t)\}\end{bmatrix}$$

$$=\left[\{h_j(L\Delta t)\}\quad\{h_j((L+1)\Delta t)\}\quad\{h_j((L+N_t-1)\Delta t)\}\right]$$

$$[B_T][h_j]=[h'_j]$$

The PolyReference Complex Exponential Method (PRCE)

$$[B_T][h_j] = [h'_j]$$

Considering for each response location *j=1,…,p:*

$$[B_T]\begin{bmatrix}[h_1] & [h_2] & \cdots & [h_p]\end{bmatrix} = \begin{bmatrix}[h'_1] & [h'_2] & \cdots & [h'_p]\end{bmatrix}$$

$$[B_T][h_T] = [h'_T] \Rightarrow$$

$$[B_T] = [h'_T][h_T]^T\left([h_T][h_T]^T\right)^{-1}$$

Knowing the coefficient matrix [B], we must now determine [V]

The PolyReference Complex Exponential Method (PRCE)

$$[\beta_0]+[\beta_1][W][V]+[\beta_2][W][V]^2+\cdots+[\beta_L][W][V]^L=[0],$$

$$\sum_{k=0}^{L}[\beta_k][W][V]^k=0$$

$$\sum_{k=0}^{L}[\beta_k][W][V]^k\begin{Bmatrix}1\\0\\\vdots\\0\end{Bmatrix}=\sum_{k=0}^{L}[\beta_k]\left(e^{s_1\Delta t}\right)^k\{W_1\}=\{0\}$$

$$\sum_{k=0}^{L}[\beta_k][W][V]^k\begin{Bmatrix}0\\1\\\vdots\\0\end{Bmatrix}=\sum_{k=0}^{L}[\beta_k]\left(e^{s_2\Delta t}\right)^k\{W_2\}=\{0\}$$

$$\Rightarrow\left[\sum_{k=0}^{L}[\beta_k]V_r^{\,k}\right]\{W_r\}=\{0\}$$

$$\vdots$$

$$\sum_{k=0}^{L}[\beta_k][W][V]^k\begin{Bmatrix}0\\1\\\vdots\\0\end{Bmatrix}=\sum_{k=0}^{L}[\beta_k]\left(e^{s_{2N}\Delta t}\right)^k\{W_{2N}\}=\{0\}$$

The PolyReference Complex Exponential Method (PRCE)

$$\left[\sum_{k=0}^{L}[\beta_k]\,V_r^{\,k}\right]\{W_r\}=\{0\}$$

$$\left[[\beta_0]+[\beta_1]\,[V]+[\beta_2]\,[V]^2+\cdots+[\beta_{L-1}]\,[V]^{L-1}\right]\{W_r\}=-[V]^L\,\{W_r\}$$

$$\{z_0\}=\{W_r\}$$

$$\{z_1\}=V_r\,\{W_r\}=V_r\,\{z_0\}$$

$$\{z_2\}=V_r^{\,2}\,\{W_r\}=V_r\,\{z_1\}$$

$$\vdots$$

$$\{z_{L-1}\}=V_r^{\,L-1}\,\{W_r\}=V_r\,\{z_{L-2}\}$$

$$\{z_L\}=V_r^{\,L}\,\{W_r\}=V_r\,\{z_{L-1}\}$$

$$\Rightarrow\quad [\beta_0]\{z_0\}+[\beta_1]\{z_1\}+\cdots$$

$$\cdots+[\beta_{L-1}]\{z_{L-1}\}=-V_r\,\{z_{L-1}\}$$

The PolyReference Complex Exponential Method (PRCE)

An standard eigenvalue problem to obtain V_r

$$\begin{bmatrix} -[\beta_{L-1}] & -[\beta_{L-2}] & \cdots & -[\beta_1] & -[\beta_0] \\ [I] & [0] & \cdots & [0] & [0] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ [0] & [0] & \cdots & [I] & [0] \end{bmatrix} \begin{Bmatrix} \{z_{L-1}\} \\ \{z_{L-2}\} \\ \vdots \\ \{z_1\} \\ \{z_0\} \end{Bmatrix} = V_r \begin{Bmatrix} \{z_{L-1}\} \\ \{z_{L-2}\} \\ \vdots \\ \{z_1\} \\ \{z_0\} \end{Bmatrix}$$

The eigenvetors z_0 correspond to W_r

The PolyReference Complex Exponential Method (PRCE)

$$\{h_j(k\,\Delta t)\} = [W][V]^k\{A_{j1}\}, \quad k = 0,1,\ldots,L \qquad \{h_j(k\,\Delta t)\} = \begin{Bmatrix} h_{j1}(k\,\Delta t) \\ h_{j2}(k\,\Delta t) \\ \vdots \\ h_{jq}(k\,\Delta t) \end{Bmatrix}$$

$$\begin{Bmatrix} \{h_j(0)\} \\ \{h_j(\Delta t)\} \\ \vdots \\ \{h_j(L\,\Delta t)\} \end{Bmatrix} = \begin{bmatrix} [W] \\ [W][V]^1 \\ \vdots \\ [W][V]^L \end{bmatrix} \{A_{j1}\} \text{ or } \{H_j\} = [W_V]\{A_{j1}\}$$

$$\{A_{j1}\} = [W_V]^+\{H_j\}$$

- The residue calculation is repeated for all measured points, *j=1,2,…,p*.

The PolyReference Complex Exponential Method (PRCE)

- The method provide more accurate modal representation of the structure.
- It can determine multiple roots or closely spaced modes.
- Shortcomings:
  - Sensitive to nonlinearities and any lack of reciprocity in frequency responses,
  - Some difficulties in analyzing structures with more than 5% viscous damping.

Global Analysis in Time Domain

(Ibrahim Time Domain Method)

Global Analysis in Time Domain (Ibrahim Time Domain Method)

- The basic concept is to obtain a unique set of modal parameters from a set of vibration measurements:
  - Scaled (mass normalized) mode shapes when the force is known,
  - Un-scaled mode shapes when the force is not measured.

Ibrahim Time Domain Method

$$x_i(t) = \sum_{r=1}^{2m} \psi_{ir} e^{s_r t}$$

$$\begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_q) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_q) \\ \vdots & \vdots & \cdots & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_q) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1,2m} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,2m} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{n,2m} \end{bmatrix} \times \begin{bmatrix} e^{s_1 t_1} & \cdots & \cdots & e^{s_1 t_q} \\ e^{s_2 t_1} & \cdots & \cdots & e^{s_2 t_q} \\ \vdots & \cdots & \cdots & \vdots \\ e^{s_{2m} t_1} & \cdots & \cdots & e^{s_{2m} t_q} \end{bmatrix}$$

$$[X] = [\Psi] \times [\Lambda]$$

Ibrahim Time Domain Method

- A 2nd set of eqns:

$$x_i(t_l + \Delta t) = \sum_{r=1}^{2m} \psi_{ir} e^{s_r(t_l + \Delta t)}$$

$$= \sum_{r=1}^{2m} \left(\psi_{ir} e^{s_r \Delta t}\right) e^{s_r t_l} = \sum_{r=1}^{2m} \hat{\psi}_{ir} e^{s_r t_l}$$

$$\left\lfloor \hat{X} \right\rfloor = \left\lfloor \hat{\Psi} \right\rfloor \times [\Lambda]$$

Ibrahim Time Domain Method

$$[A]\times[\Psi]=[\hat{\Psi}]$$

$$\left.\begin{aligned}[X]&=[\Psi]\times[\Lambda]\\ [\hat{X}]&=[\hat{\Psi}]\times[\Lambda]\end{aligned}\right\}\Rightarrow[A]\times[X]=[\hat{X}]$$

$$[A]=[\hat{X}]\times[X]^{+}$$

Ibrahim Time Domain Method

$$[A]\{\psi_r\} = e^{s_r \Delta t}\{\psi_r\}$$

- Eigenvectors of matrix [A] are the mode shapes,
- The natural frequencies and damping ratios are obtained from eigenvalues of [A].

Modal Testing

(Lecture 19)

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Derivation of Mathematical Models

- Spatial Models (mass, stiffness, damping)
  - Needs measurement of most of the modes
  - Requires measurement in many DOFs
- Response Models (FRF)
  - Needs measurement in frequency range of interest
  - Requires measurement in selected DOFs
- Modal Models (natural frequencies and mode shapes)
  - Needs measurement of only one mode
  - Requires measurement in handful of DOFs

Derivation of Mathematical Models

- Modal Models
  - Requirements to construct Modal Models
  - Refinement of Modal Model
    - Conversion to real modes
    - Compatibility of DOFs
      - Reduction
      - Expansion
- Response Models
  - FRF
  - Transmissibility
  - Base Excitation

Requirement to construct Modal Models

- Minimum requirements
  - One column in case of fixed excitation or
  - One row when response is measured at a fixed point.

$$\begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1i} & \cdots & \boxed{H_{1j}} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2i} & \cdots & H_{2j} & \cdots & H_{2n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ H_{i1} & H_{i2} & \cdots & H_{ii} & \cdots & H_{ij} & \cdots & H_{in} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ H_{j1} & H_{j2} & \cdots & H_{ji} & \cdots & H_{jj} & \cdots & H_{jn} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{ni} & \cdots & H_{nj} & \cdots & H_{nn} \end{bmatrix}$$

Requirement to construct Modal Models

- Proof:

$$\alpha_{mn} = \frac{X_m(\omega)}{F_n(\omega)}$$

$$= \frac{X_m(\omega)}{F_i(\omega)} \times \frac{X_i(\omega)}{F_n(\omega)} \times \frac{F_i(\omega)}{X_i(\omega)}$$

$$= \frac{\alpha_{mi}\alpha_{ni}}{\alpha_{ii}}$$

Requirement to construct Modal Models

- Several additional elements of FRF or even columns are measured to:
  - Replace poor data,
  - To provide checks
  - Modes have not been missed

$$
\begin{bmatrix}
H_{11} & H_{12} & \cdots & H_{1i} & \cdots & H_{1j} & \cdots & H_{1n} \\
H_{21} & H_{22} & \cdots & H_{2i} & \cdots & H_{2j} & \cdots & H_{2n} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
H_{i1} & H_{i2} & \cdots & H_{ii} & \cdots & H_{ij} & \cdots & H_{in} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
H_{j1} & H_{j2} & \cdots & H_{ji} & \cdots & H_{jj} & \cdots & H_{jn} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
H_{n1} & H_{n2} & \cdots & H_{ni} & \cdots & H_{nj} & \cdots & H_{nn}
\end{bmatrix}
$$

Refinement of Modal Models

- Complex to real conversion:
  - Taking the modulus of each element and assigning a phase of 0 or 180.
  - Finding a real mode with maximum projection to the measured one:

$$\max \frac{\left|\phi_R^T \phi_C\right|}{\left|\phi_R\right|\left|\phi_C\right|}$$

  - Multi point excitation (Asher's method)

Compatibility of DOFs

- Employment of the measured modes in updating/modification of analytical models requires the compatibility of DOFs.
- There are two approaches in compatibility excursive:
  - Analytical model reduction
  - Expansion of measured modes

Reduction of Analytical Model (Guyan Reduction)

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}$$

$$x_2 = -K_{22}^{-1} K_{12}^T x_1$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} I \\ -K_{22}^{-1} K_{12}^T \end{bmatrix} \{x_1\}, \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [T]\{x_1\}$$

$$T^T \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} T\{x_1\} = \{f_1\}$$

Dynamic Model Reduction

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}\right)\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0.$$

$$\phi_2 = -\left(K_{22} - \omega^2 M_{22}\right)^{-1}\left(K_{12}^T - \omega^2 M_{12}^T\right)\phi_1$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} I \\ -\left(K_{22} - \omega^2 M_{22}\right)^{-1}\left(K_{12}^T - \omega^2 M_{12}^T\right) \end{bmatrix}\{\phi_1\}, \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = [T]\{\phi_1\}$$

$$T^T\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}\right)T\{\phi_1\} = 0.$$

Expansion of Models

- In order to compare analytical model with the measured modal data on may expand the measured data by:
  - Geometric interpolation using spline functions
  - Using analytical model spatial model
  - Using analytical model modal model

Expansion in Spatial Domain

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}\right)\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0.$$

$$\phi_2 = -\left(K_{22} - \omega^2 M_{22}\right)^{-1}\left(K_{12}^T - \omega^2 M_{12}^T\right)\phi_1$$

Expansion in Modal Domain

$$\Phi = \Phi_0 R^T$$

Assuming Mass Matrix is correct

$$Sol:$$

$$\min_R \left\| \tilde{\Phi} R - \tilde{\Phi}_0 \right\| \qquad st: \; R^T R = I.$$

$$\tilde{\Phi}^T \tilde{\Phi}_0 = U \Sigma V^T \Rightarrow R = V U^T,$$

Response Model

- Frequency response functions

$$[H] = [\Phi]\left[\left(\lambda_r^2 - \omega^2\right)^{-1}\right][\Phi]^T$$

- Transmissibilities

$$T_{jk}(\omega) = \frac{X_j e^{i\omega t}}{X_k e^{i\omega t}}, \qquad {}_iT_{jk}(\omega) = \frac{H_{ji}(\omega)}{H_{ki}(\omega)}$$

Transmissibility Plots

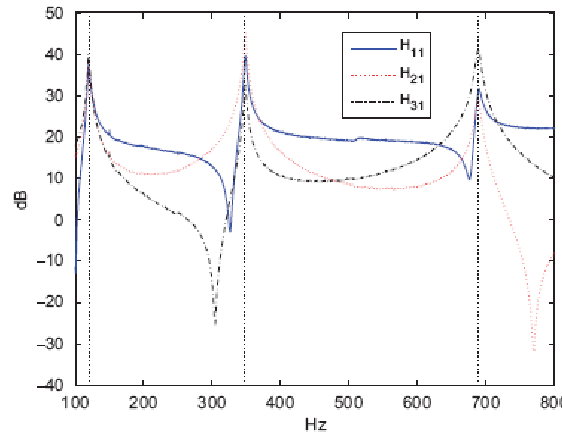


Fig. 2. Frequency response data.

$$T_{ij}(\omega) = X_i(\omega)/X_j(\omega)$$

T^k_{21} and T^k_{32} with $k = 1, \ldots, 3$.

force locations

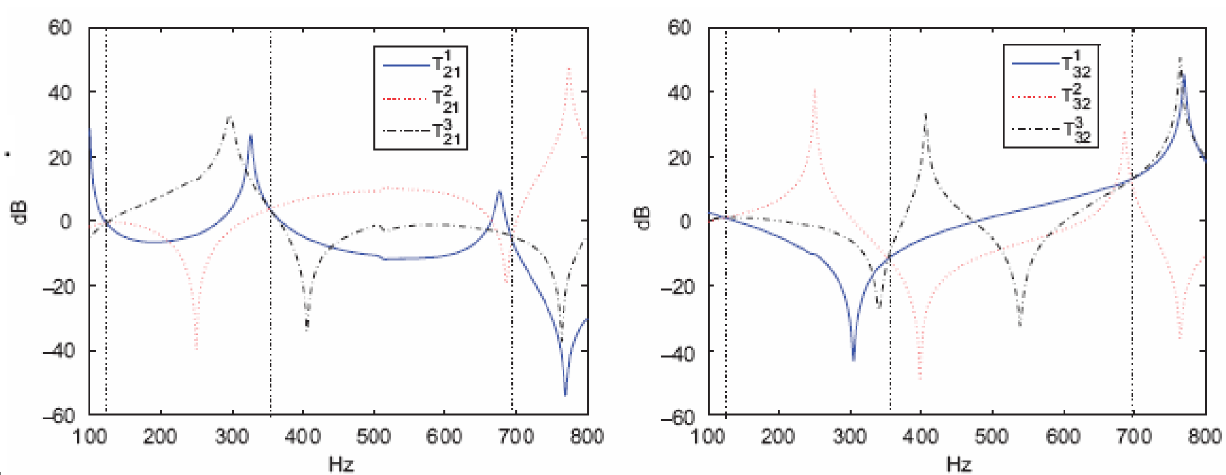


Fig. 3. Transmissibilities.

Derivation of Mathemat

Response Model

- The amplitude's peaks of the transmissibilities do not correspond with the resonant frequencies.
- Transmissibilities cross each other at the resonant frequencies (becomes independent of the location of the input)

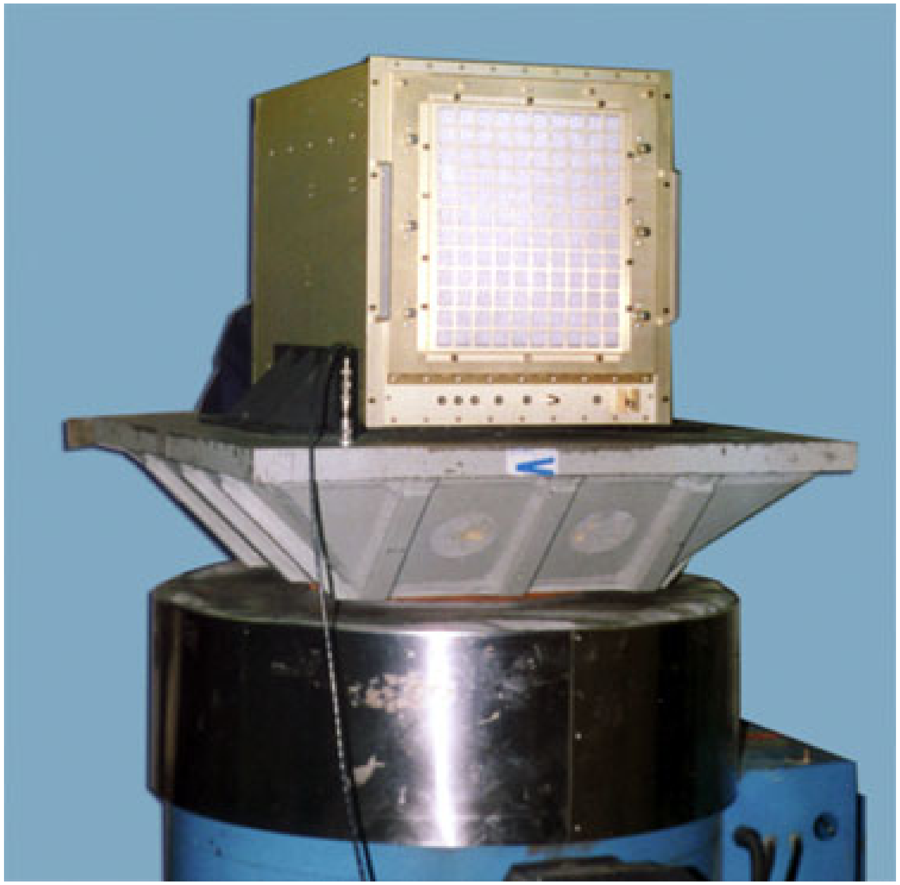
$$ {}_iT_{jk}(\omega) = \frac{H_{ji}(\omega)}{H_{ki}(\omega)} = \frac{\sum_r \frac{\phi_{jr}\phi_{ir}}{\omega_r^2 - \omega^2}}{\sum_r \frac{\phi_{kr}\phi_{ir}}{\omega_r^2 - \omega^2}} $$

$$ {}_iT_{jk}(\omega)\Big|_{\omega\approx\omega_r} \approx \frac{\phi_{jr}}{\phi_{kr}} $$

Base Excitation

- An application area of transmissibility.
- Input is measured as response at the drive point.

$$\{x\} = \{x_{rel}\} + x_{ref}\begin{Bmatrix}1\\1\\\vdots\\1\end{Bmatrix}$$



Base Excitation

$$[M]\{\ddot{x}_{rel}\}+[K]\{x_{rel}\}=-\ddot{x}_{ref}[M]\{g\},\ \{g\}=\begin{Bmatrix}1\\1\\\vdots\\1\end{Bmatrix}.$$

$$[H(\omega)]^{-1}\left(\{X\}-x_{ref}\{g\}\right)=\omega^2 x_{ref}[M]\{g\},$$

$$or\ \ \frac{\left(\{X\}-x_{ref}\{g\}\right)}{\omega^2 x_{ref}}=[H(\omega)][M]\{g\}.$$

Base Excitation

$$\frac{\left(\{X\} - x_{ref}\{g\}\right)}{\omega^2 x_{ref}} = \left([H(\omega)]\right)\left([M]\{g\}\right),$$

$$\{Q\} = \frac{\left(\{X\} - x_{ref}\{g\}\right)}{\omega^2 x_{ref}}, \{u\} = [M]\{g\},$$

$$Q_i(\omega) = \sum_j \sum_r \frac{\phi_{ir}\phi_{jr}}{\omega_r^2 - \omega^2} u_j$$

Spatial Models

$$[M] = [\Phi]^{-T}[\Phi]^{-1}$$

$$[K] = [\Phi]^{-T}[\lambda_r][\Phi]^{-1}$$

Modal Testing

(Lecture 20)

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Derivation of Mathematical Models

- Introduction
- Equation Error Method (Sec. 6.3.6 page 456)
  - Identification of Rod FE Model
  - Parameter Identification
- Solution of Over-determined set of Equations
- Solution of Under-determined set of Equations
- Error Analysis

Introduction

- Construction of Spatial Model from modal data:

$$K = \Phi^{-T} \Lambda \Phi^{-1}, M = \Phi^{-T} \Phi^{-1}, C = \Phi^{-T} \Gamma \Phi^{-1}$$

- Modal model must be complete:
  - All modes must be present
  - Mode shapes are measured in all DOF's
- Measurement of complete Modal Model is impractical.

Introduction

- Alternative methods are required to construct the spatial model from
  - incomplete and
  - noisy measured modes.
- The difficulty with incompleteness is removed by reducing the number of unknowns in spatial model.
- The noise effects are removed by averaging.

Equation Error Method

- We have some information regarding the spatial model format:
  - Symmetry
  - Pattern of zeros
  - …
- We may incorporate these information into the identification procedure and reconstruct the spatial model.

Equation Error Method

- In this method the eigen problem is rearranged to obtain the spatial model:

$$K\Phi - M\Phi\Lambda = 0.$$

$$\Rightarrow \begin{bmatrix} \Phi^T & \Lambda\Phi^T \end{bmatrix} \begin{Bmatrix} K \\ M \end{Bmatrix} = 0$$

- The DOF's of measured modes must be compatible with the DOF's of spatial model.

Rearrangement Example:

$$\left(\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega_r^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right)\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0,$$

$or:$

$$\begin{bmatrix} \phi_1 & \phi_1 - \phi_2 & -\omega_r^2 \phi_1 & 0 \\ 0 & \phi_2 - \phi_1 & 0 & -\omega_r^2 \phi_2 \end{bmatrix}\begin{Bmatrix} k_1 \\ k_2 \\ m_1 \\ m_2 \end{Bmatrix} = 0.$$

Identification of A Rod FE Model

- Consider a fixed-free rod with *n* elements.
- The mass and stiffness matrices are:

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \cdots & \cdots & \cdots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}$$

$$M = diag(m_1, m_2, \cdots, m_{n-1}, m_n)$$

Identification of A Rod FE Model

- The equilibrium state at modes *r* and *s* are:

$$(K - \lambda_r M)\phi_r = 0.$$

$$(K - \lambda_s M)\phi_s = 0.$$

- The last rows of equilibrium state equations are:

$$-k_n \phi_{r,n-1} + (k_n - \lambda_r m_n)\phi_{r,n} = 0,$$

$$-k_n \phi_{s,n-1} + (k_n - \lambda_r m_n)\phi_{s,n} = 0.$$

Identification of A Rod FE Model

$$\begin{bmatrix} \phi_{r,n} - \phi_{r,n-1} & -\lambda_r \phi_{r,n} \\ \phi_{s,n} - \phi_{s,n-1} & -\lambda_s \phi_{s,n} \end{bmatrix} \begin{Bmatrix} k_n \\ m_n \end{Bmatrix} = 0.$$

$$k_n \neq 0, m_n \neq 0 \Rightarrow \begin{cases} \lambda_s = \lambda_r \dfrac{\phi_{r,n}\left(\phi_{s,n} - \phi_{s,n-1}\right)}{\phi_{s,n}\left(\phi_{r,n} - \phi_{r,n-1}\right)} \\ \dfrac{k_n}{m_n} = \dfrac{\lambda_r \phi_{r,n}}{\phi_{r,n} - \phi_{r,n-1}} \end{cases}$$

Identification of A Rod FE Model

- From other rows one obtains:

$$\frac{k_1}{m_n}, \frac{k_2}{m_n}, \ldots, \frac{k_{n-1}}{m_n}, \frac{m_1}{m_n}, \frac{m_2}{m_n}, \ldots, \frac{m_{n-1}}{m_n}$$

- Using total mass information m_m is obtained:

$$m_{total} = m_n \left(1 + \sum_{l=1}^{n-1} \frac{m_l}{m_n} \right)$$

Identification of A Rod FE Model

- Only two modes and one natural frequency are required to construct the mass and stiffness matrices.
- More details can be found in:
  - GML Gladwell, YM Ram, ”Constructing Finite Element Model of a Vibrating Rod”, Journal of Sound and Vibration, **169**,229-237,1994.

Parameter Identification

- In a general case the mass and stiffness matrices are parameterized and are obtained by rearranging:
  - Equation of motion in modal domain
  - Orthogonality requirements,
  - etc.

Parameter Identification

$$Parameterization:$$

$$K = K(k_1, k_2, \ldots, k_n), M = M(m_1, m_2, \ldots, m_n).$$

$$EOM:$$

$$K\Phi - M\Phi\Lambda = 0, \Phi^T M\Phi = I, \Phi^T K\Phi = \Lambda.$$

$$Extras:$$

$$K\Phi_R = 0, \qquad \Phi_R^T M\Phi_R \Rightarrow m, I_{xx}, I_{xy}, \ldots$$

Parameter Identification

$\text{Re}-\text{arrangement}:$

$$Ax = b$$

$$A = A(\Phi, \Phi_R, \Lambda), b = b(\Lambda, m_{total}, I_{xx}, \ldots)$$

$$x = x(k_1, k_2, \ldots, k_n, m_1, m_2, \ldots, m_n)$$

$$A \Rightarrow full \quad rank(n \times m) \Rightarrow \begin{cases} m = n \rightarrow x = A^{-1}b \\ m > n \rightarrow Under\det er\min ed \\ m < n \rightarrow Over\det er\min ed \end{cases}$$

Solution of underdetermined case

$$\min\|x - x_0\|, ST : Ax = b$$

or

$$\min\|\Delta x\|, ST : A\Delta x = b - Ax_0 = \bar{b}$$

Solution :

$$\min\left(\Delta x^T \Delta x - 2\left(\Delta x^T A^T - \bar{b}^T\right)\lambda\right) \Rightarrow 2\Delta x - 2A^T\lambda = 0.$$

$$\Rightarrow AA^T\lambda = \bar{b} \Rightarrow \lambda = \left(AA^T\right)^{-1}\bar{b} \Rightarrow \Delta x = A^T\left(AA^T\right)^{-1}\bar{b}$$

Solution of Over-determined set of Equations

$$Ax = b, \qquad m < n$$

$$Ax - b = \varepsilon$$

$$Assume \Rightarrow E[\varepsilon_i] = 0., E[\varepsilon\varepsilon_j] = \delta_{ij}\sigma$$

$$Solution \Rightarrow \min\|\varepsilon\| = \min \varepsilon^T \varepsilon$$

$$\varepsilon^T \varepsilon = x^T A^T A x - 2x^T A^T b + b^T b$$

$$\frac{\partial(\varepsilon^T \varepsilon)}{\partial x} = 2A^T A x - 2A^T b \Rightarrow \bar{x} = (A^T A)^{-1} A^T b$$

Error Analysis

$$b = Ax + \varepsilon$$

$$E\left[\left(A^T A\right)^{-1} A^T b\right] = E[x] + E\left[\left(A^T A\right)^{-1} A^T \varepsilon\right]$$

$$E[\bar{x}] = E[x] + E\left[\left(A^T A\right)^{-1} A^T \varepsilon\right]$$

$$as \quad m \to \infty, E[\varepsilon] \to 0 \Rightarrow E[\bar{x}] = E[x]$$

$$If\ A \to \quad noisy \to E\left[A^T \varepsilon\right] \neq 0 \Rightarrow E[\bar{x}] \neq E[x]$$

Example:

- The parameters to be updated are the 10 stiffness and 6 masses
- The measured data consists of the 1st three natural frequencies and mode shapes (added with uniformly distributed random noise)

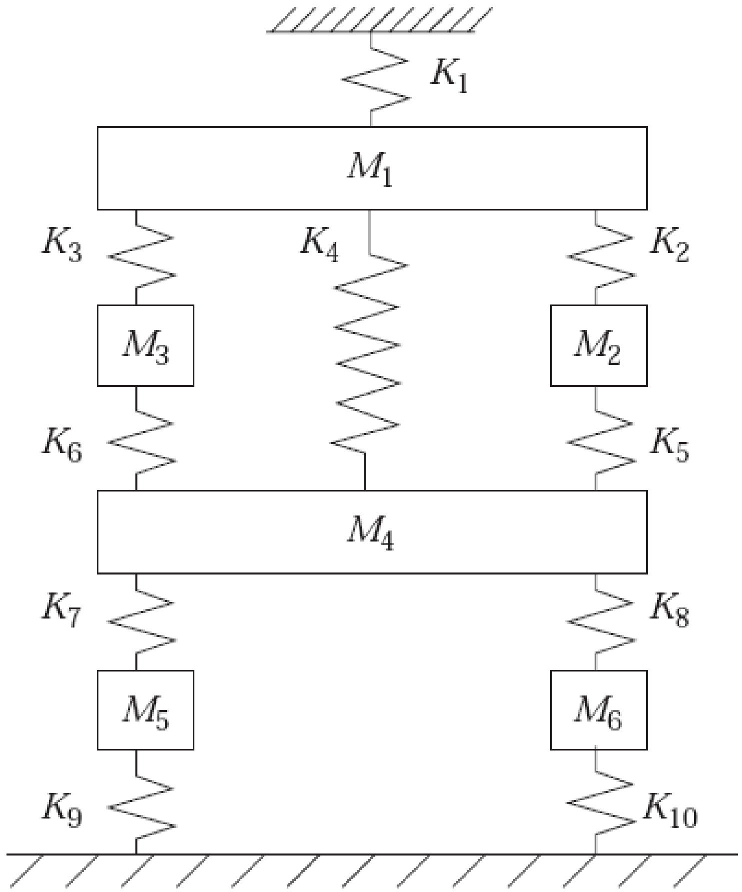

Figure 1. Numerical spring–mass model.

Example:

- Eigenvalue equations arearrangment:
  - 31 equations (3*6 equations for each eigenvector term, 2*6 symmetric orthogonality equations, and 1 total mass equation)
  - 16 parameters and
  - The terms in *A* and *b* contain noisy data.

$$Parameters: x = \{k_1, k_2, \ldots, k_{10}, m_1, m_2, \ldots, m_6\},$$

$$EOM: K\Phi - M\Phi\Lambda = 0, \Phi^T M\Phi = I, \Phi^T K\Phi = \Lambda.$$

$$Extras: \phi_R^T M \phi_R = m.$$

Example

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	m_1	m_2	m_3	m_4	m_5	m_6
Exact	1000	1250	1500	10000	1000	1000	5000	7000	1000	1000	1	0.2	0.1	1	0.1	0.1
S/N= 100	1041	1266	1518	10008	1006	1008	9914	-414	2035	-63	1	0.2	0.1	0.99	0.2	-0.01
S/N= 20	954	1243	1502	9844	1160	1011	-1024	21352	-83	2323	0.97	0.21	0.1	0.96	-0.02	0.28

Regularized Solution

- Ahmadian, Mottershead, and Friswell, REGULARISATION METHODS FOR FINITE ELEMENT MODEL UPDATING, *Mechanical Systems and Signal Processing* (1998) 12(1),47-64

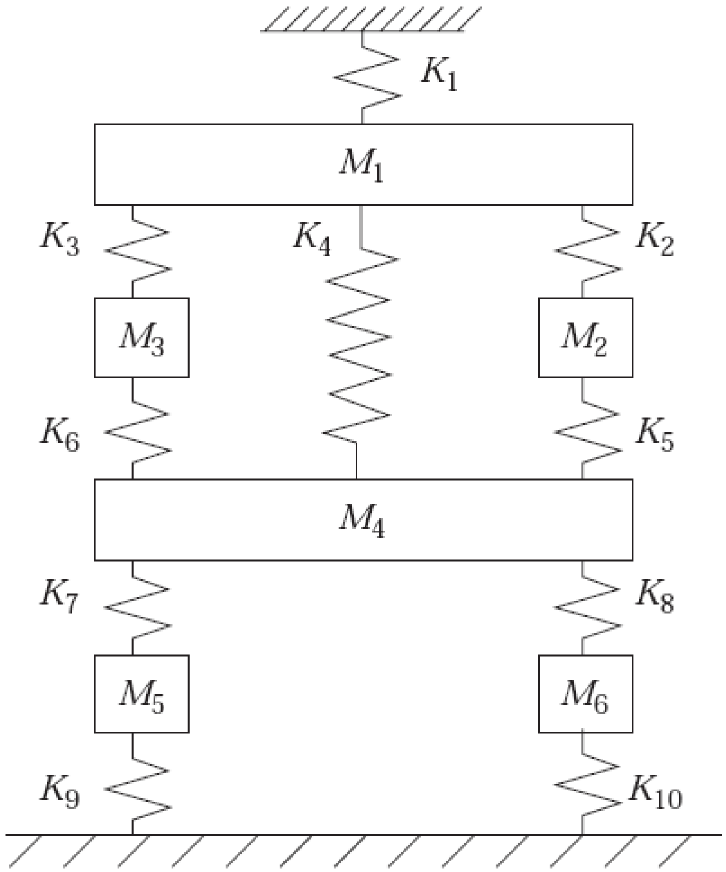

Figure 1. Numerical spring–mass model.

Home Work 3

- Develop a procedure to construct the FE model of a fixed-free beam from minimum modes.
  - How many modes are required to obtain EI an m of each element?
  - Add some noise to the modes and try to reconstruct the model.
  - Investigate the correlated noise effects?